Linear Algebra Notes

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1 Basic Definitions

1.1 Fields and Vectors

We begin with some of the most basic definitions in linear algebra.

Definition 1.1. *A field is a set F with elements $0, 1 \in F$ along with operations $+, \cdot : F \times F \to F$ (i.e. operations taking two field elements and returning one) satisfying the following axioms $\forall x, y, z \in F$*

1. $x + y = y + x$ *(Commutativity)*
2. $(x + y) + z = x + (y + z)$ *(Associativity)*
3. $x + 0 = x$ *(Additive Identity)*
4. $\exists(-x) \in F$ *such that* $x + (-x) = 0$ *(Additive Inverse)*
5. $x \cdot y = y \cdot x$ *(Commutativity)*
6. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ *(Associativity)*
7. $1 \cdot x = x$ *(Multiplicative Identity)*
8. $\exists x^{-1} \in F$ *such that* $x \cdot x^{-1} = 1$ *(Multiplicative Inverse)*
9. $x \cdot (y + z) = x \cdot y + x \cdot z$ *(Distributivity)*

Notation 1.2. *By convention, the $\cdot$ is dropped for multiplication, so $x \cdot y$ is written xy.*

Essentially, fields are a generalization of the properties that make rational numbers so nice to work with, with the elements $0, 1$ being generalizations of those same elements in $\mathbb{Q}$.

Example 1.3. *$\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are all fields. Some more exotic fields include $\mathbb{F}_p$, the finite field of integers modulo $p \in \mathbb{N}$, where p is prime.*

With fields in hand, we can now define vectors (whose axioms borrow their names from those in Definition 3.2).

Definition 1.4. *A vector space V over a field F is a set with an element $\vec{0} \in V$ and operations $+ : V \times V \to V$ and $\cdot : F \times V \to V$ satisfying the following axioms $\forall \vec{u}, \vec{v}, \vec{w} \in V, x, y \in F$*

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
3. $\vec{u} + \vec{0} = \vec{u}$
4. $\exists(-\vec{u}) \in F$ *such that* $\vec{u} + (-\vec{u}) = \vec{0}$
5. $(x \cdot y) \cdot \vec{u} = x \cdot (y \cdot \vec{u})$
6. $(x + y) \cdot \vec{u} = x \cdot \vec{u} + y \cdot \vec{u}$
7. $1 \cdot \vec{u} = \vec{u}$
8. $x \cdot (\vec{u} + \vec{v}) = x \cdot \vec{u} + x \cdot \vec{v}$

Notation 1.5. *Note that we are essentially overloading the operations $+, \cdot$, with different definitions for their actions on fields and vectors. The operations being used is generally clear from context, and the same convention is used for both. We will use an overhead arrow to indicate vectors for the rest of this document. Other common notations include $\mathbf{v}$ and $\underline{v}$.*

Example 1.6. *$\mathbb{Q}^n, \mathbb{R}^n, \mathbb{C}^n$ are all vector spaces, where $n \in \mathbb{N}$. However, we could define these vector spaces to be over the field $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. For example, we could regard $\mathbb{R}^n$ as a vector space over $\mathbb{Q}$, or as one over $\mathbb{R}$.*

The above example is not particularly interesting, but we can construct more exotic examples.

Example 1.7. *The set of all functions from $\mathbb{R} \to \mathbb{R}$ is a vector space over the field $\mathbb{R}$. We define addition by $(f+g)(x) = f(x) + g(x)$ and scalar multiplication by $(af)(x) = a(f(x))$.*

1.2 Subspaces and Coordinates

Definition 1.8. *A set $U \subseteq V$ is called a subspace of V if it is closed under addition and scalar multiplication. That is, if $\vec{u}, \vec{v} \in U$ and $x \in F$, then $\vec{u} + \vec{v} \in U$ and $x\vec{u} \in U$.*

Example 1.9. *$\{(x, 0) \mid x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$, but is only a subspace of $\mathbb{C}^2$ is we take $\mathbb{C}^2$ to be over the field $\mathbb{Q}$ or $\mathbb{R}$.*

The above example illustrates an important point: while we define vector spaces abstractly, they're essentially impossible to work with without a coordinate system. Thus, we will now build up the necessary machinery to move into coordinate systems.

Definition 1.10. *A set $U = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\} \subseteq V$ is called linearly independent if $\sum_{i=1}^{n} \alpha_i \vec{u}_i = \vec{0}$ implies that $\alpha_i = 0$, where $\alpha_i \in F$. A set that is not linearly independent is called linearly dependent.*

There is a way to extend the above definition, and the rest of this section, to infinite sets. If you're interested in that, see section 4.

Example 1.11. *The set $\{(1,0,0), (0,1,0)\} \in \mathbb{R}^3$ is linearly independent.*

Definition 1.12. *The span of a set $U \subseteq V$ is the set*

$$\mathrm{Span}_F(U) = \left\{ \sum_{i=1}^{n} \alpha_i \vec{u}_i \mid \alpha_i \in F \right\} \tag{1}$$

that is, any vector that can be constructed by a linear sum of vectors in U.

Example 1.13. $\mathrm{Span}_{\mathbb{C}}(\{(1,0)\}) = \{(x, 0) \mid x \in \mathbb{C}\}$.

Notation 1.14. *If $X = \mathrm{Span}_F(U)$, we say that U spans X.*

Combining these two notions, we can finally define the basis of a vector space.

Definition 1.15. *A set $X \subseteq V$ is called a basis if it is linearly independent and $\mathrm{Span}_F(X) = V$.*

The most basic example of this would be the standard basis in $\mathbb{R}^n$, and every vector space has a basis[1]. However, a basis is useless for our notion of coordinates unless it results in unique coordinates. We prove this now.

Theorem 1.16. *Let X be a basis for V. Then for any $\vec{v} \in V$ there exist unique $\alpha_i \in F$ such that $\vec{v} = \sum_{i=1}^{n} \alpha_i \vec{x}_i$, where $\vec{x}_i \in X$.*

Proof. Suppose $\alpha_i, \beta_i \in F$ exist such that $\vec{v} = \sum_{i=1}^{n} \alpha_i \vec{x}_i = \sum_{i=1}^{n} \beta_i \vec{x}_i$. Then $\sum_{i=1}^{n} (\alpha_i - \beta_i)\vec{x}_i = \vec{0}$. Since X is a basis, it is linearly independent, and hence by Definition 1.10 we must have that $\alpha_i - \beta_i = 0$, so $\alpha_i = \beta_i$. □

It's not essential that you understand any of the proofs in this document, the writer of this section is just a bored math student and wanted to prove things.

[1] Assuming the Axiom of Choice, which physics does as far as I'm aware

Notation 1.17. *The unique α_i for any given $\vec{v} \in V$ relative to a basis X are referred to as $\vec{v}$'s coordinates in that basis. These coordinates are what allow us to consider many arbitrary vector spaces as being essentially identical to F^n, in a sense that will be made precise in section 2.*

A final useful concept is that of dimension. This requires the following theorem, which we will not prove.

Theorem 1.18. *Suppose that $X, Y \subseteq V$ are bases of V. Then $|X| = |Y|$, that is both sets have the same size.*

Definition 1.19. *The dimension of a vector space V is the size of its basis set.*

Example 1.20. *$\mathbb{R}^3$ has a standard basis of size 3, and so has dimension 3.*

There are many other useful properties of linearly independent sets, we list some below.

Theorem 1.21. *Let V be a vector space over a field F. Then the following all hold.*

1. *Any subset of a linearly independent set is linearly independent.*
2. *Any proper superset of a linearly dependent set is linearly dependent.*
3. *Any set with size larger than the basis of V is linearly dependent.*

2 Linear Mappings

2.1 Basics of Mappings

To start this section, we cover some preliminaries of maps which may be unfamiliar to some. Note that map is essentially just a *fancy*[2] word for function.

Definition 2.1. *Let X, Y be sets. A map $f : X \to Y$ is called injective if for all $a, b \in X$, $f(a) = f(b)$ if and only if $a = b$. The map is called surjective if for all $y \in Y$, there exists some $x \in X$ such that $f(x) = y$. A map that is injective and surjective is called bijective.*

Are we done with notation? Of course not, mathematics just cannot resist more notation. We denote by 1 the map which satisfies $1(x) = x$ for all $x \in X$, essentially the map that does nothing. This is called the identity map.[3] We say that a map is invertible if there exists another map $g : Y \to X$ such that $g \circ f = f \circ g = 1$.[4] We denote this inverse f^{-1} when it exists. With this, we can finally get to the point.

Theorem 2.2. *A function $f : X \to Y$ is invertible if and only if it is bijective.*

Proof. First, suppose that f is invertible, with inverse f^{-1}. If $f(x) = f(y)$, where $x, y \in X$, then this would imply that $f^{-1}(f(x)) = f^{-1}(f(y))$, so $x = y$. Hence f is injective. Pick any $y \in Y$. Then $f(f^{-1}(y)) = y$, so $\exists x = f^{-1}(y) \in X$ such that $f(x) = y$. Hence, f is surjective and therefore bijective. Now, suppose that f is bijective. Pick any $y \in Y$. Then there exists a unique $x \in X$ such that $f(x) = y$. Hence, the map $g : Y \to X$ given by $g(y) = x$ is well-defined. A quick check shows that g is indeed the inverse of f. □

[2] Mathematicians love to be fancy.

[3] In fact, the same identity as in Example 1.7

[4] We abuse notation here, the two compositions produce identity maps on X and Y respectively, so they're technically not equal

Before we leave, we need one more piece of notation.

Definition 2.3. *The image of a map $f : X \to Y$ is given by*

$$\mathrm{Im}(f) = \{y \in Y \mid \exists x \in X, f(x) = y\}$$

2.2 Linear Maps

We now come to the central concept of linear algebra.

Definition 2.4. *A linear map is a map $f : V \to W$ between two vector spaces over a common field F which satisfies, for all $\vec{u}, \vec{v} \in V, \alpha \in F$:*

- *Additivity: $f(\vec{u} + \vec{v}) = f(\vec{u}) + f(\vec{v})$.*
- *Homogeneity: $f(\alpha\vec{v}) = \alpha f(\vec{v})$.*

This can be simplified and written as one property $f(\alpha\vec{u} + \vec{v}) = \alpha f(\vec{u}) + f(\vec{v})$.

We denote by $\mathcal{L}(V, W)$ the set of all linear maps from V to W. Consider the following examples:

Example 2.5. *The zero map (denoted by 0) takes any element of a vector space V to the additive identity of another vector space W. Specifically, $0 \in \mathcal{L}(V, W)$ is defined by*

$$0\vec{v} = 0, \text{ for any } \vec{v} \in V.$$

similarly, we have

Example 2.6. *The identity element (denoted by I) which maps any $\vec{v} \in V$ to itself. Specifically, $I \in \mathcal{L}(V, V)$ is defined as*

$$I\vec{v} = \vec{v}.$$

If you wanted to define a linear map in general, you'd need to define its action on each vector in the vector space. This of course is not feasible, making the following theorem immensely useful.

Theorem 2.7. *Let $X \subseteq V$ be a basis. Then any linear map $f : V \to W$ can be uniquely specified by its action on X. Furthermore, any choice of image for the basis produces a valid linear map.*

Proof. Pick any $\vec{v} \in V$. Then $\exists \alpha_i \in F$ such that $\vec{v} = \sum_{i=1}^{n} \alpha_i \vec{x}_i$. Thus, we get that

$$f(\vec{v}) = f\Big(\sum_{i=1}^{n} \alpha_i \vec{x}_i\Big) = \sum_{i=1}^{n} \alpha_i f(\vec{x}_i)$$

Since each $f(\vec{x}_i)$ is uniquely defined, so is $f(\vec{v})$. The unique definition follows from noting that since X is linearly independent, no vector in X can be constructed as a sum of the others, and thus the choice of the other $f(\vec{x}_j)$ places no restriction on possible choices of $f(\vec{x}_i)$[5]. □

We can also multiply two linear operators to get a composition of operations. Consider $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$. We define $ST \in \mathcal{L}(U, W)$ by

$$(ST)(\vec{v}) = S(T\vec{v}), \text{ for } \vec{v} \in U.$$

This is the same as the composition of two functions.

Whenever it makes sense[6], the product of two linear map have the following properties:

[5] Yes, I hand waved.

[6] We need to verify that the product of two linear map is also a linear map.

- Associativity: $T_1(T_2T_3) = (T_1T_2)T_3$.
- Identity: $TI = T$ and $IT = T$ whenever $T \in \mathcal{L}(V, W)$.
- Distributive property: $(S_1 + S_2)T = S_1T + S_2T$ and $S(T_1 + T_2) = ST_1 + ST_2$ whenever $T, T_1, T_2 \in \mathcal{L}(U, V)$ and $S, S_1, S_2 \in \mathcal{L}(V, W)$.

2.3 Null Spaces

For a $T \in \mathcal{L}(V, W)$ we define the null space/kernel (denoted $\ker(T)$) as

$$\ker(T) = \{\vec{v} \in V |\ T\vec{v} = \vec{0}\}.$$

Example 2.8. *Let $T \in \mathcal{L}(V, W)$ be the operation of differentiation. Then any element $\vec{v} \in \ker(T)$ satisfies $T\vec{v} = \vec{v}'$. The only vectors that have a zero derivative are constant vectors, hence the null space of T equals the set of constant vectors.*

Now we start building up towards one of the most important theorems in this chapter: the Rank-Nullity theorem.

Proposition 2.1. *If $T \in \mathcal{L}(V, W)$, then $\ker(T)$ is a subspace of V.*

Proof. Suppose $T \in \mathcal{L}(V, W)$, then by additivity,

$$T(\vec{0}) = T(\vec{0} + \vec{0}) = T(\vec{0}) + T(\vec{0}) \implies T(\vec{0}) = \vec{0},$$

and thus $\vec{0} \in \ker(T)$. For some $\vec{u}, \vec{v} \in \ker(T)$, we have

$$T(\vec{u} + \vec{v}) = T\vec{u} + T\vec{v} = 0 \implies \vec{u} + \vec{v} \in \ker(T),$$

and thus $\ker(T)$ is closed under addition. If $\vec{v} \in \ker(T)$ and $a \in F$, then

$$T(a\vec{v}) = aT(\vec{v}) = a\vec{0} = \vec{0} \implies a\vec{v} \in \ker(T),$$

and thus $\ker(T)$ is closed under scalar multiplication. Since $\ker(T)$ contains $\vec{0}$ and is closed under both addition and scalar multiplication, $\ker(T)$ is a subspace of V. □

Now we continue with the next proposition[7].

Proposition 2.2. *When $T \in \mathcal{L}(V, W)$, T is injective if and only if $\ker(T) = \{\vec{0}\}$.*

Proof. Suppose that T is injective. We want to show that $\ker(T) = \{\vec{0}\}$. Consider some $\vec{v} \in \ker(T)$; we have

$$T\vec{v} = \vec{0} = T(\vec{0}) \xrightarrow{T \text{ is injective}} \vec{v} = \vec{0}.$$

Thus, $\ker(T) = \{\vec{0}\}$. To prove the other direction of the implication, suppose that null $\ker(T) = \{\vec{0}\}$. For some $\vec{u}, \vec{v} \in V$, suppose that $T\vec{u} = T\vec{v}$. Then,

$$\vec{0} = T\vec{u} - T\vec{v} = T(\vec{u} - \vec{v}) \implies \vec{u} - \vec{v} \in \ker(T).$$

So we have $\vec{u} - \vec{v} = \vec{0}$ which gives us $\vec{u} = \vec{v}$, proving that T is injective. □

[7] Bear with me for a bit; I understand if these propositions seem a bit random, but we are building towards what I believe are the most important results of this chapter.

It is then easy to show that, similarly, for $T \in \mathcal{L}(V, W)$, $\text{Im}(T)$ is a subspace of W[8].
Now, we finally start showing what we have been building up to this entire section:

Theorem 2.9. *If V is finite dimensional and $T \in \mathcal{L}(V, W)$, then $\text{Im}(T)$ is a finite-dimensional subspace of W, and*

$$\dim(V) = \dim(\ker(T)) + \dim(\text{Im}(T)).$$

where dim *denotes the dimension of a space.*

Proof. Let V be a finite-dimensional vector space, and $T \in \mathcal{L}(V, W)$. Let $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m$ be a basis of $\ker(T)$. We can extend this set of linearly independent vectors to a linearly independent set $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m, \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$, which forms a basis for V. So $\dim(V) = m + n$, and the only thing remaining to show is that $\text{Im}(T)$ is finite dimensional, and $\dim(\text{Im}(T)) = n$. We can do this by showing that $(T\vec{w}_1, T\vec{w}_2, \ldots, T\vec{w}_n)$ is a basis for image T. Let $\vec{v} \in V$. Since $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m, \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ spans V, we can write

$$\vec{v} = a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_m\vec{u}_m + b_1\vec{w}_1 + b_2\vec{w}_2 + \cdots + b_n\vec{w}_n.$$

where $a_i, b_i \in F$. Applying T on both sides, we get

$$\begin{aligned} T\vec{v} =& T(a_1\vec{u}_1) + T(a_2\vec{u}_2) + \cdots + T(a_m\vec{u}_m) + T(b_1\vec{w}_1) + T(b_2\vec{w}_2) + \cdots + T(b_n\vec{w}_n) \\ =& T(b_1\vec{w}_1) + T(b_2\vec{w}_2) + \cdots + T(b_n\vec{w}_n), \end{aligned} \tag{2}$$

where $T(a_1\vec{u}_1) + T(a_2\vec{u}_2) + \cdots + T(a_m\vec{u}_m) = \vec{0}$ because $\vec{u}_i \in \ker(T)$. We also know from eq. (2) that $T\vec{w}_1, T\vec{w}_2, \ldots, T\vec{w}_n$ span $\text{Im}(T)$, which implies that $\text{Im}(T)$ is finite-dimensional. To show that these are also linearly independent consider that for $c_i \in F$, we have that if

$$\begin{aligned} \vec{0} =& c_1T(\vec{w}_1) + c_2T(\vec{w})_2 + \cdots + c_nT(\vec{w}_n) \\ =& T(c_1\vec{w}_1 + c_2\vec{w}_2 + \cdots + c_n\vec{w}_n), \end{aligned}$$

then $c_1\vec{w}_1 + c_2\vec{w}_2 + \cdots + c_n\vec{w}_n \in \ker(T)$. Thus, for $d_i \in F$ we have

$$c_1\vec{w}_1 + c_2\vec{w}_2 + \cdots + c_n\vec{w}_n = d_1\vec{u}_1 + d_2\vec{u}_2 + \cdots + d_m\vec{u}_m.$$

Since $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m, \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ are linearly independent, we know that all $c_i = d_j = 0$, and so we have shown that

$$\dim(V) = \dim(\ker(T)) + \dim(\text{Im}(T)),$$

□

2.4 Matrices

When we think of linear algebra, we think of matrices. But what really even is a matrix?

Definition 2.10. *For natural numbers m, n a matrix is defined as a rectangular array having m rows and n columns which has the structure:*

$$\begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix},$$

where the first index of the element denotes the row, and the second index denotes the column, e.g., a_{43} is the element of the fourth row and third column.

[8]It is also a good exercise in one of the more important skills when dealing with vector spaces: proving that something is a subspace of given vector space.

While this is a definition for a matrix, it only defines the structure of a matrix; it can seem quite random and hard to reconcile with anything that we have done up till this point. We now work towards improving our intuition on what a matrix is, and connecting it to linear maps.

Let $T \in \mathcal{L}(V, W)$. Then, for every $k = 1, 2, \ldots, n$, we can write $T\vec{v}_k$ as a unique linear combination of the basis of W, i.e.,

$$T\vec{v}_k = a_{1,k}\vec{w}_1 + a_{2,k}\vec{w}_2 + \cdots + a_{m,k}\vec{w}_m, \text{ where } a_{j,k} \in F.$$

Here we can clearly observe that the a's determine the linear map T. The m, n matrix formed by these a's is called the **matrix** of T, and it is denoted by $\mathcal{M}(T, (v_1, v_2, \ldots, v_n), (w_1, w_2, \ldots, w_m))$. In cases where the bases are obvious, we just write $\mathcal{M}(T)$. Thus, we can think of matrices as representations of linear maps. However, this raises the question of whether they behave as linear maps; how do they add? How do they multiply? These are questions we will answer now.

Definition 2.11. *We define matrix addition as*

$$\begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \ldots & b_{mn} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & \ldots & a_{1n} + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \ldots & a_{mn} + b_{mn} \end{pmatrix}.$$

Only matrices that are of the same size can be added, i.e., an $m \times n$ matrix can only be added to another $m \times n$ matrix. This can be represented in terms of linear maps as

$$\mathcal{M}(T + S) = \mathcal{M}(T) + \mathcal{M}(S),$$

for some $T, S \in \mathcal{L}(V, W)$.

Definition 2.12. *We define the multiplication of a scalar and a matrix as*

$$c\begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix} = \begin{pmatrix} ca_{11} & \ldots & ca_{1n} \\ \vdots & \ddots & \vdots \\ ca_{m1} & \ldots & ca_{mn} \end{pmatrix}.$$

In terms of linear maps, this is

$$\mathcal{M}(cT) = c\mathcal{M}(T),$$

for $c \in F$ and $T \in \mathcal{L}(V, W)$.

Now that we have defined both addition and scalar multiplication, we should be able to show that there exists a vector space $\text{Mat}(m, n, F)$ of all $m \times n$ matrices with elements all from F[9].

All that is left now is to consider what our matrix would look like if we had a composition of two linear maps. Consider three vector spaces V, W, and U having bases $(v_1, v_2, \ldots, v_n)$, $(w_1, w_2, \ldots, w_m)$, and $(u_1, u_2, \ldots, u_p)$ respectively. Let $S : U \to V$ and $T : V \to W$ be linear maps; the composition TS is a linear map from $U \to W$. How can we find $\mathcal{M}(TS)$ from $\mathcal{M}(T)$ and $\mathcal{M}(S)$? We know that in the case of linear maps $T \circ S = TS$, so the cleanest solution would be $\mathcal{M}(TS) = \mathcal{M}(T)\mathcal{M}(S)$, however we have not yet defined matrix multiplication. Hence, we will define matrix multiplication such that it forces the above equation to be true.

[9] A good starting point is the additive identity: the $m \times n$ matrix with all elements being 0.

Definition 2.13. *Let*

$$\mathcal{M}(T) = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \text{ and } \mathcal{M}(S) = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{np} \end{pmatrix}.$$

For $k = \{1, 2, \dots, p\}$, we have

$$\begin{aligned} TS\vec{u}_k =& T\left(\sum_{r=1}^{n} b_{r,k}\vec{v}_r\right) \\ =& \sum_{r=1}^{n} b_{r,k} T(\vec{v}_r) \\ =& \sum_{r=1}^{n} b_{r,k} \sum_{j=1}^{m} a_{j,r}\vec{w}_j \\ =& \sum_{j=1}^{m} \left(\sum_{r=1}^{n} \underbrace{a_{j,r}b_{r,k}}_{\alpha}\right) \vec{w}_j. \end{aligned}$$

Thus, $\mathcal{M}(TS)$ is the $m \times p$ matrix whose entry in row j column k is given by α. This definition of matrix multiplication only holds when the number of columns of the first matrix is equal to the number of rows of the second matrix. Matrix multiplication is also clearly not commutative.

2.5 Invertibility

Here, we show some important results on inverting linear maps.

Proposition 2.3. *If a linear map $T : U \to V$ is invertible, then its inverse is a linear map.*

Proof. Pick any $\vec{v}_1, \vec{v}_2 \in V$ and $\alpha \in F$. Since T is invertible, there exist unique $\vec{u}_1, \vec{u}_2 \in U$ such that $T\vec{u}_1 = \vec{v}_1, T\vec{u}_2 = \vec{v}_2$. Thus, we get that

$$T^{-1}(\alpha\vec{v}_1 + \vec{v}_2) = T^{-1}(\alpha T(\vec{u}_1) + T\vec{u}_2) = T^{-1}(T(\alpha\vec{u}_1 + \vec{u}_2)) = \alpha\vec{u}_1 + \vec{u}_2 = \alpha T^{-1}(\vec{v}_1) + T^{-1}(\vec{v}_2)$$

as required. □

Definition 2.14. *Two vector spaces are called isomorphic if there exists an invertible linear map from one vector space to the other.*

The point of the above definition is that isomorphic vector spaces are essentially the same, just presented in different forms. This is exemplified in the following theorem.

Theorem 2.15. *Two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension.*

Proof. To begin, consider V and W are finite-dimensional vector spaces that are isomorphic to one another. Let there exist an invertible map $T \in \mathcal{L}(V, W)$. Since T is invertible $\ker(T) = \{\vec{0}\}$ and $\text{Im}(T) = W$. Thus, $\dim(\ker(T)) = 0$ and $\dim(\text{Im}(T)) = \dim(W)$. Then from the Rank-Nullity theorem,

$$\begin{aligned} \dim(V) =& \dim(\ker(T)) + \dim(\text{Im}(T)) \\ =& \dim(W). \end{aligned}$$

To prove the other direction of the implication, suppose that V and W are finite-dimensional vector spaces having the same dimension. Let $(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$ be the basis of V and $(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n)$ be the basis of W. For $a_i \in F$ let the linear map $T \in \mathcal{L}(V, W)$ be defined as

$$T(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n) = a_1\vec{w}_1 + a_2\vec{w}_2 + \cdots + a_n\vec{w}_n.$$

T is clearly surjective since $(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n)$ spans W, and since $(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n)$ are linearly independent, T is injective. As we have shown before, since T is bijective, it is invertible, and V and W are isomorphic. □

Corollary 2.15.1. *Every n dimensional vector space is isomorphic to F^n.*

Given a vector space V having basis $(v_1, v_2, \ldots, v_n)$ and vector space W having basis $(w_1, w_2, \ldots, w_m)$, for each $T \in \mathcal{L}(V, W)$ there exists a matrix $\mathcal{M}(T) \in \text{Mat}(m, n, F)$. This follows clearly from the section on matrices, and we have also shown before that $\mathcal{M}(T)$ is a linear map. Similar to T is $\mathcal{M}(T)$ also invertible when it is bijective?

Proposition 2.4. *Suppose $(v_1, v_2, \ldots, v_n)$ is a basis of V and $(w_1, w_2, \ldots, w_m)$ is a basis of W, then there exists an invertible map $\mathcal{M}$ between $\mathcal{L}(V, W)$ and Mat(m, n, F).*

Proof. We require to show that $\mathcal{M}$ is bijective. To begin, let us prove injectivity. For some $T \in \mathcal{L}(V, W)$, if $\mathcal{M}(T) = 0$, then $Tv_k = 0$ for all $k \in \{1, 2, \ldots, n\}$. Since $(v_1, v_2, \ldots, v_n)$ is a basis of V, implying $T = 0$. It follows clearly from proposition 2.2. that $\mathcal{M}(T) = 0$. To prove surjectivity, consider a matrix

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix},$$

which lies in $\text{Mat}(m, n, F)$. Let $T \in \mathcal{L}(V, W)$ such that

$$Tv_k = \sum_{j=1}^{m} a_{j,k} w_j \text{ for } k \in \{1, 2, \ldots, n\}.$$

Clearly, $\mathcal{M}(T) = A$, and thus image of $\mathcal{M}(T)$ is $\text{Mat}(m, n, F)$, as required. □

2.5.1 Operators

We start with a quick definition.

Definition 2.16. *A linear map from a vector space onto itself is called an **operator**. The set of all operators on a vector space is denoted by $\mathcal{L}(V, V) = \mathcal{L}(V)$.*

We have shown before that for a linear map between two vector spaces to be invertible, we require it to be bijective. Is this the same for an operator? Perhaps, since the map is from a vector space to itself, only injectivity or surjectivity might be enough since they could imply each other. This is in fact true, and we shall prove this now.

Theorem 2.17. *Suppose V is finite dimensional. If $T \in \mathcal{L}(V)$, then the following are equivalent statements:*

(a) T is invertible.

(b) T is injective.

(c) T is surjective.

Proof. Suppose $T \in \mathcal{L}(V)$. We have shown before that (a) clearly implies (b). Suppose T is injective, i.e., $\ker(T) = \{\vec{0}\}$. Thus, from Rank-Nullity theorem

$$\begin{aligned}\dim(V) &= \dim(\ker(T)) + \dim(\mathrm{Im}(T)) \\ &= \dim(\mathrm{Im}(T)),\end{aligned}$$

which implies that $\mathrm{Im}(T) = V$, proving that T is surjective. Hence, $(b) \implies (c)$.

Suppose T is surjective, i.e., $\mathrm{Im}(T) = V$. Once again, by Rank-Nullity theorem,

$$\begin{aligned}\dim(\ker(T)) &= \dim(V) - \dim(\mathrm{Im}(T)) \\ &= 0,\end{aligned}$$

which implies that T is injective, and by extension, T is bijective. Thus, T is invertible, and $(c) \implies (a)$. □

2.6 Norms and Traces

Before we finish this section, there are two more operations that we wish to talk about.

Definition 2.18. *Let V be a vector space over $\mathbb{C}$. A norm $\|\cdot\| : V \to \mathbb{R}^+$ is a function which satisfies the following three properties*

1. *For all $\vec{x}, \vec{y} \in V$, $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ (triangle inequality).*
2. *For all $\vec{x} \in V, \alpha \in \mathbb{C}$, $\|\alpha\vec{x}\| = |\alpha|\|\vec{x}\|$.*
3. *$\|\vec{x}\| = 0$ if and only if $\vec{x} = \vec{0}$.*

Example 2.19. *The most common example of this is the euclidean norm on $\mathbb{R}^n$, given by*

$$\|\vec{x}\| = \sqrt{\sum_{j=1}^{n} x_j^2}$$

Norms exist to give us a notion of size in a vector space. In fact, this notion of size can be extended to $\mathcal{L}(V)$ as well.

Definition 2.20. *Let V be a vector space with a norm, and let $T \in \mathcal{L}(V)$. Define the function $f_T : V \backslash \{\vec{0}\} \to \mathbb{R}^+$ by the following equation*

$$f_T(\vec{x}) = \frac{\|T\vec{x}\|}{\|\vec{x}\|}$$

It turns out that exactly one of the following must be true.

1. *There exists some smallest $\alpha \in \mathbb{R}^+$ such that $f_T(\vec{x}) \leq \alpha$ always. In this case, we say that $\|T\| = \alpha$.*
2. *f_T is unbounded, in which case we say that $\|T\| = \infty$.*

For our purposes we only really need this definition in section 4, but it's interesting nonetheless and does come up in more advanced quantum mechanics. Finally, we'll bring up the following definition which is much more important in quantum mechanics.

Definition 2.21. *The trace of $A \in F^{n\times n}$ is defined by*

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} A_{ii}$$

that is the sum of the diagonal elements of A.

It turns out that this definition can be extended to any linear operator on a finite dimensional vector space. In order to show this, we'll need the following Lemmas, the first of which we'll prove.

Lemma 2.22. *For any $A, B \in F^{n\times n}$, $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.*

Proof.

$$\mathrm{Tr}(AB) = \sum_{i=1}^{n}(AB)_{ii} = \sum_{i=1}^{n}\Big(\sum_{j=1}^{n} A_{ij}B_{ji}\Big) = \sum_{j=1}^{n}\Big(\sum_{i=1}^{n} B_{ji}A_{ij}\Big) = \sum_{j=1}^{n}(BA)_{jj} = \mathrm{Tr}(BA)$$

□

Lemma 2.23. *Let V be an n-dimensional vector space and $T \in \mathcal{L}(V)$. Let $A, B \in F^{n\times n}$ be the matrix representations of T in two different bases. Then there exists a matrix $Q \in F^{n\times n}$ such $B = QAQ^{-1}$.*

The matrix Q in the second lemma is called a change of basis matrix. Using these, we can make our definition

Definition 2.24. *Let V be an n-dimensional vector space and $T \in \mathcal{L}(V)$. The trace of T is defined by*

$$\mathrm{Tr}(T) = \sum_{i=1}^{n} \mathcal{M}(T)_{ii}$$

that is the sum of the diagonal elements of the matrix representation of T.

Note that our above definition is well-defined since it doesn't matter which basis we choose to take a matrix representation in. Indeed, suppose that A, B are two different matrix representations of T. Then by Lemma 2.23, there exists some matrix Q such that $B = QAQ^{-1}$. By Lemma 2.22, we have that $\mathrm{Tr}(B) = \mathrm{Tr}\big(QAQ^{-1}\big) = \mathrm{Tr}\big(Q^{-1}QA\big) = \mathrm{Tr}(A)$, as required.

3 Dual Spaces and Diagonalization

3.1 Eigenvectors and Eigenvalues

Definition 3.1. *Let A be a matrix. The vector $\vec{v} \neq \vec{0}$ is an eigenvector of A iff there exists a constant λ such that*

$$A\vec{v} = \lambda\vec{v}$$

The constant λ is called an eigenvalue of A.

Definition 3.2. *The matrix A is diagonalizable iff there exists an invertible matrix M and a diagonal matrix D such that*

$$A = MDM^{-1}$$

Note that diagonalizations are not unique.

Briefly, for an n-dimensional vector space, a matrix A is diagonalizable if it has a basis of eigenvectors; that is, we can find n linearly independent eigenvectors. Then M is constructed by taking the eigenvectors as columns and putting their corresponding eigenvalue in the nth column of D. As a result, diagonalizing a diagonalizable matrix corresponds exactly to finding all of its eigenvectors and eigenvalues, which also provide important information about how it behaves.

Practically, diagonalization also allows us to view the "spectrum" of the matrix, or how it changes given inputs as a mapping. The action of a matrix A on an eigenvector $\vec{v}$ is just to scale it by λ, and when we can construct a basis of eigenvectors of A, we can understand how every "component" of an arbitrary vector $\vec{u} \in V$ changes under A by expanding it in this basis.

Definition 3.3. *Let V be a vector space over $\mathbb{C}$ (or a subset of $\mathbb{C}$) and let $\vec{u}, \vec{v} \in V$, $k \in \mathbb{C}$. An inner product is a mapping*

$$\langle \vec{u}, \vec{v} \rangle : V \times V \to \mathbb{C}$$

with the following properties:

1. $\langle \vec{u}, \vec{u} \rangle \geq 0$ *with equality only when* $\vec{u} = 0$
2. $\langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{u} \rangle}$
3. $k\langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle = \langle \vec{u}, k\vec{v} + \vec{w} \rangle$

Inner products generalize the idea of the dot product in $\mathbb{R}^n$ to more general vector spaces and situations. They provide additional structure to a vector space that is often useful, so vector spaces equipped with an inner product $\langle \cdot, \cdot \rangle$ are separately called *inner product spaces*. This generalized "inner product" also motivates the following generalization of orthogonality:

Definition 3.4. *Let $\vec{u}, \vec{v} \in V$ be two vectors in an inner product space V. $\vec{u}$ and $\vec{v}$ are said to be orthogonal if*

$$\langle \vec{u}, \vec{v} \rangle = 0$$

Definition 3.5. *Let V be an n-dimensional vector space. A basis $\{\vec{u}_1, \vec{u}_2 \dots \vec{u}_n\}$ is said to be orthonormal if*

$$\langle \vec{u}_i, \vec{u}_j \rangle = \delta_{ij}$$

where δ_{ij} is the Kronecker delta function, which satisfies

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}$$

Orthogonality is defined based on the inner product, which means that an inner product must be specified when discussing it. In analogy to $\mathbb{R}^n$, we also have the following result:

Lemma 3.6. *Let V be an n-dimensional vector space with an orthonormal basis $\{\vec{u}_1, \vec{u}_2 \dots \vec{u}_n\}$. For every $\vec{v} \in V$,*

$$\vec{v} = \sum_{i=1}^{n} \langle \vec{u}_i, \vec{v} \rangle \vec{u}_i$$

As a result, performing calculations in an orthonormal basis is often more convenient, especially when inner products are relevant. A natural question is whether every vector space has an orthonormal basis, and we can show that it does by constructing an orthogonal basis out of any arbitrary basis. To show this, we first introduce the notation of projections:

Definition 3.7. *Let $\vec{u}, \vec{v} \in V$ be two vectors in some vector space V. The projection of $\vec{u}$ onto $\vec{v}$ is the vector*

$$proj_{\vec{v}}(\vec{u}) = \frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v}$$

Lemma 3.8. *Let V be a vector space with a basis $\{\vec{u}_1, \vec{u}_2 \dots \vec{u}_n\}$. Then the basis*

$$\begin{aligned}
\vec{v}_1 &= \vec{u}_1 \\
\vec{v}_2 &= \vec{u}_2 - proj_{\vec{v_1}}(\vec{u_2}) \\
\vec{v}_3 &= \vec{u}_3 - proj_{\vec{v}_1}(\vec{u}_3) - proj_{\vec{v}_2}(\vec{u_3}) \\
&\vdots \\
\vec{v}_n &= \vec{u}_n - \sum_{i=1}^{n-1} proj_{\vec{v}_i}(\vec{u}_n)
\end{aligned}$$

is orthogonal (and each vector can be normalized). The process of computing this orthonormal basis is called the Gram-Schmidt process or Gram-Schmidt orthogonalization.

The most important result of this section is the *spectral theorem* for symmetric and Hermitian matrices, which provides conditions for matrices being diagonalizable with an orthonormal basis of eigenvectors, which allows us to work with orthonormal eigenvectors of an arbitrary Hermitian operator. To do this, we introduce the following definitions:

Definition 3.9. *The transpose of a matrix A is the matrix A^T whose i, jth entry is a_{ji}. Informally, this is A but with the rows and columns switched. A matrix A is called symmetric if $A^T = A$.*

Definition 3.10. *A matrix H is said to be Hermitian if $H^\dagger = H$, where $H^\dagger$ is the conjugate transpose of H.*

Definition 3.11. *A matrix Q is called orthogonal if it satisfies $QQ^T = Q^TQ = I$. Equivalently, the rows and columns of Q form orthonormal bases. Analogously, a matrix U is unitary if it satisfies $UU^\dagger = U^\dagger U = I$.*

A matrix being symmetric/Hermitian is a sufficient condition to have an orthonormal basis of eigenvectors:

Theorem 3.12. *Let S be a symmetric matrix. There exists an orthogonal matrix Q and a diagonal matrix D such that*

$$S = QDQ^{-1} = QDQ^T$$

Similarly, we have for Hermitian matrices that:

Theorem 3.13. *If H is a Hermitian matrix, there exists a unitary matrix U and a diagonal matrix D such that*

$$H = UDU^\dagger$$

These theorems tell us that whenever we talk about Hermitian or symmetric matrices, we can automatically assume that they are diagonalizable, and that we can construct an orthonormal basis of eigenvectors. As a result, we can always choose an orthonormal basis (where inner products simplify significantly).

3.2 Dual Spaces

It turns out that one can see the inner product as an example of a more general phenomena : linear functionals.

Definition 3.14. *Let V be a vector space over a field F. A linear function $f : V \to F$ is called a linear functional. The set V^* of all linear functionals on V is a vector space in itself, and referred to as the dual space of V.*

As an example, let's look at how this connects to the inner product.

Example 3.15. *Pick any vector $\vec{v} \in V$. Then the function $f(\vec{u}) = \langle \vec{v}, \vec{u} \rangle$, where $\vec{u} \in V$, is a linear functional in V^*.*

It's also worth considering how the basis of the dual space relates to the basis of a vector space.

Definition 3.16. *Pick any vector $\vec{v} \in V$. Let $B \subset V$ be a basis of V such that $\vec{v} \in B$. The dual of $\vec{v}$ relative to this basis, denoted $\vec{v'} \in V^*$, is the unique linear functional defined by, for any $\vec{u} \in B$*

$$\vec{v'}(\vec{u}) = \begin{cases} 1 & \vec{v} = \vec{u} \\ 0 & \vec{v} \neq \vec{u} \end{cases} \tag{3}$$

Note that the dual of a vector is inextricably tied to the basis we've chosen, choosing a different basis with the same vector can lead to a different dual. For any given basis B, the set B^* of the dual of each element of B is called the *dual basis* of V.

Theorem 3.17. *Let B be a basis for V. Then B^* is a basis for V^*.*

Proof. First, we'll show that it's linearly independent. Let $B = \{\vec{b}_i\}_{i=1}^n$. Suppose that $\sum_{i=1}^n \alpha_i \vec{b'}_i = \vec{0'}$. Then for any $\vec{b}_j$, $\left(\sum_{i=1}^n \alpha_i \vec{b'}_i\right)(\vec{b}_j) = 0$, so $\alpha_j = 0$. Thus, B^* is linearly independent. Next, we show that it's spanning. Pick any $f \in V^*$. Then it's clear that $f = \sum_{i=1}^n f(\vec{b}_i)\vec{b'}_i$ (since it's fully defined by its action on the basis), making B^* spanning. □

4 Additional Topics

4.1 Infinitesimal Rotations

Let's start with an important generalization : you can take derivatives of functions between any two normed vector spaces.

Definition 4.1. *Let V, W be normed vector spaces and let $f : V \to W$ be a function. For any given $\vec{x}_0 \in V$, we say that f is differentiable at $\vec{x}_0$ if there exists some linear operator $L_{\vec{x}_0} : V \to W$ such that*

$$\lim_{\vec{x} \to \vec{x}_0} \frac{\|f(\vec{x}) - f(\vec{x}_0) - L_{\vec{x}_0}(\vec{x} - \vec{x}_0)\|}{\|\vec{x} - \vec{x}_0\|} = 0 \tag{4}$$

If it exists, we call $L_{\vec{x}_0}$ the differential of f at $\vec{x}_0$. This is usually denoted $df_{\vec{x}_0}$.

This is rather abstract, so let's take a look at how it connects to our normal definition of the derivative.

Example 4.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\mathbb{R}$. We note that $\mathbb{R}$ is a vector space over $\mathbb{R}$ with norm $\|x\| = |x|$, and that linear operators take the form $T(x) = ax$ for some $a \in \mathbb{R}$. Now, pick any $x_0 \in \mathbb{R}$. Then we need for*

$$\lim_{x \to x_0} \frac{|f(x) - f(x_0) - L_{x_0}(x - x_0)|}{|x - x_0|} = 0$$

We may write that $L_{x_0}(y) = ay$ for some $a \in \mathbb{R}$, and hence get that

$$\frac{|f(x) - f(x_0) - L_{x_0}(x - x_0)|}{|x - x_0|} = \Big| \frac{f(x) - f(x_0)}{x - x_0} - \frac{a(x - x_0)}{x - x_0} \Big| = \Big| \frac{f(x) - f(x_0)}{x - x_0} - a \Big|$$

Since $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$, we can see that $a = f'(x_0)$. So $df_{x_0} = f'(x_0)$, that is the differential is just evaluating the derivative!

Speaking of derivatives, we can generalize their definition too.

Definition 4.3. *Let V, W be normed vector spaces and let $f : V \to W$ be a differentiable function. Then the map Df which is defined by $Df(\vec{x}) = df_{\vec{x}}$ is the derivative of f.*

Note : For those of you who have taken multivariable calculus, the above definitions are in fact equivalent to the ones you would've seen in class.

Now that we've gone over of these preliminaries, let's get to the point. Consider a rotation by θ in $\mathbb{R}^2$. This can be represented by the linear operation

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

There's something to note here : R can also be viewed as a function $R : \mathbb{R} \to \mathbb{R}^{2\times 2}$. Both of these are normed vector spaces, and hence we can find the differential of R. $dR \in \mathbb{R}^{2\times 2}$, so we can do this component-by-component. Indeed, we can note that

$$\begin{aligned} \lim_{\theta \to \theta_0} \Big| \frac{\cos\theta - \cos\theta_0 - a(\theta - \theta_0)}{\theta - \theta_0} \Big| &= \lim_{\theta \to \theta_0} \Big| \frac{(1 - \frac{\theta^2}{2} + \dots) - (1 - \frac{\theta_0^2}{2} + \dots) - a(\theta - \theta_0)}{\theta - \theta_0} \Big| \\ &= \lim_{\theta \to \theta_0} \Big| \frac{(-\frac{\theta^2}{2} + \dots) - (-\frac{\theta_0^2}{2} + \dots) - a(\theta - \theta_0)}{\theta - \theta_0} \Big| \end{aligned}$$

But of course any terms of order two or higher in $\theta - \theta_0$, and hence in θ or θ_0, will go to zero faster than $\theta - \theta_0$, so this simplifies down to $\lim_{\theta \to \theta_0} \left| \frac{a(\theta - \theta_0)}{\theta - \theta_0} \right| = |a|$. Thus, $a = 0$. Similarly, we can get that

$$\begin{aligned} \lim_{\theta \to \theta_0} \Big| \frac{\sin\theta - \sin\theta_0 - b(\theta - \theta_0)}{\theta - \theta_0} \Big| &= \lim_{\theta \to \theta_0} \Big| \frac{(\theta - \frac{\theta^3}{3!} + \dots) - (\theta_0 - \frac{\theta_0^3}{3!} + \dots) - b(\theta - \theta_0)}{\theta - \theta_0} \Big| \\ &= \lim_{\theta \to \theta_0} \Big| \frac{(\theta - \theta_0) - b(\theta - \theta_0)}{\theta - \theta_0} \Big| = |1 - b| \end{aligned}$$

so $b = 1$. Combining these results, we can get that for any $\theta_0 \in \mathbb{R}$ our differential is

$$dR_{\theta_0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

which is independent of angle. From now on, we just call this dR.

Now, let's take a step back here and notice something quite interesting. Pick any $\vec{x} \in \mathbb{R}^2$, and let $\delta\theta$ be a small rotation. We can note that

$$\lim_{\delta\theta \to 0} \frac{\|R(\delta\theta) - R(0) - dR(\delta\theta)\|}{\|\delta\theta\|} = 0$$

so it follows that since $R(0) = I$

$$\lim_{\delta\theta \to 0} \frac{\|(R(\delta\theta)\vec{x} - \vec{x} - dR(\delta\theta)\vec{x}\|}{\|\delta\theta\|} \leq \lim_{\delta\theta \to 0} \frac{\|(R(\delta\theta) - R(0) - dR(\delta\theta)\|\|\vec{x}\|}{\|\delta\theta\|} = 0$$

That is for small angles, we get

$$R(\delta\theta)\vec{x} \approx \vec{x} + dR(\delta\theta)\vec{x}$$

This is $dR(\delta\theta)$ is referred to as an infinitesimal rotation. It's essentially a very good approximation of the effects of a rotation on a vector for small angles.

Here's the really interesting thing, we can go the other way. We can define our rotation operator by saying that for small angles $\delta\theta$

$$R(\delta\theta)\vec{x} \approx \vec{x} + dR(\delta\theta)\vec{x}$$

Say we wanted to rotate through a large angle θ. We could divide this into N rotations by θ/N, computing each smaller rotation with our approximation. The approximation becomes exact as $N \to \infty$, so

$$R(\theta)\vec{x} = \lim_{N \to \infty} \Big(I + dR(\theta/N)\Big)^N \vec{x}$$

You may recognize this limit in the form $\lim_{N\to\infty} \Big(1+\theta/N\Big)^N = e^\theta$. This suggests that our rotation matrix should be $R(\theta) = e^{dR(\theta)}$. So let's answer one final question; is this the case, and if so what does it even mean?

Let's start with the question of what $e^{dR(\theta)}$ represents. One way of defining e^x is via the following formula

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

So we're going to try and extend this definition to linear operators. Say $T : V \to V$ is a linear operator on a normed vector space. Then we define that

$$e^T = \sum_{n=0}^{\infty} \frac{T^n}{n!}$$

when this sum converges. Our notion of convergence here is the same used for sums of real numbers, with the absolute value replaced with the operator norm. All of the properties of e^x can be derived from its series expansion, so it follows that these properties carry over to e^T. $dR(\theta/N)$ is a linear operator from $\mathbb{R}^2 \to \mathbb{R}^2$, and we can get using the binomial expansion that

$$\Big(I + dR(\theta/N)\Big)^N = \sum_{n=0}^{N} \binom{N}{n} dR^n \Big(\frac{\theta}{N}\Big)^n = \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} dR^n \Big(\frac{\theta}{N}\Big)^n$$

For any particular n, we get that

$$\begin{aligned}\lim_{N\to\infty}\frac{N!}{n!(N-n)!N^n} &= \lim_{N\to\infty}\frac{(N-1)\cdots(N-n+1)}{n!N^{n-1}} \\ &= \frac{1}{n!}\lim_{N\to\infty}\frac{(N-1)\cdots(N-n+1)}{N^{n-1}} = \frac{1}{n!}\end{aligned}$$

so

$$R(\theta) = \sum_{n=0}^{\infty}\frac{1}{n!}dR^n(\theta^n) = e^{dR(\theta)}$$

just like we were hoping! Finally, let's compute what this matrix actually is. Through induction, you can show that

$$dR^n = \begin{cases}\begin{pmatrix}\frac{(-1)^{n/2}}{n!} & 0\\ 0 & \frac{(-1)^{n/2}}{n!}\end{pmatrix} & \text{n even}\\ \begin{pmatrix}0 & -\frac{(-1)^{(n-1)/2}}{n!}\\ \frac{(-1)^{(n-1)/2}}{n!} & 0\end{pmatrix} & \text{n odd}\end{cases}$$

the coefficients in the even case correspond to the power series for $\cos x$, and the odd case those for $\sin x$, so it follows that

$$e^{dR(\theta)} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

meaning that we've recovered exactly what we started with!

4.2 Infinite Dimensional Vector Spaces

Let's start by taking a closer look at example 1.7 and asking a simple question : what is the basis of this vector space? It seems reasonable to say that no finite set of set of functions will do. And this is not the only case of this phenomena, consider for example the vector space of finite sequences

$$V = \left\{\{q_i\}\in\mathbb{R}^{\mathbb{N}} \mid \text{finitely many } q_i \text{ are non-zero}\right\} \tag{5}$$

This as well fails to have any finite basis. There are even physically relevant examples of this sort of vector space. Those of you familiar with some quantum may be aware that the vector space representing the state of a free particle also has no finite basis. Thus, any rigorous treatment of vector spaces cannot limit itself to considering only finite bases.

Now in order to define infinite bases, we first need to decide what it means to sum up an infinite number of basis vectors. Unfortunately, we lack any notion of distance or limits, so there is in fact no consistent useful way to define an infinite sum. Luckily, we can sidestep the problem.

Definition 4.4. *Let I be an indexing set, that is a set for which every element $i \in I$ is associated with exactly one vector $\vec{u}_i$ in the vector space V. Given any choice of $\alpha_i \in F$, the underlying field, we say that $\sum_{i\in I}\alpha_i\vec{u}_i$ is well-defined only if at most a finite number of the α_i are non-zero. When the sum is well-defined, we set it equal to the finite sum including only terms with non-zero coefficients.*

Note the important distinction here : our definition doesn't actually tell us how to sum up an infinite number of vectors, it just tells us how to write an arbitrary number of terms in our ostensibly finite sum. However, it suffices for our purposes. We can use this definition to naturally extend the definitions of span, linear independence, and basis to infinite sets, in each case only considering finite sums of vectors picked from the infinite set.

This is still very abstract, so let's go back to our example from Equation 5. We define $\{\delta_{ij}\}$ as the sequence which satisfies the following $\delta_{ij} = 1$ when $i = j$, and zero otherwise (essentially the Kroekner delta function). Then the following set is in fact a basis for V

$$B = \{\{\delta_{ij}\} \mid i \in \mathbb{N}\}$$

Since any element of V can have only finitely many non-zero elements in its sequence, we only ever need to sum up finitely many elements of B, so this works with our new definition!

The next natural question would be which of our theorems from before carry over to infinite-dimensional vector spaces. Theorem 1.16 carries over unchanged, and Theorem 1.18 carries over replacing the size of the basis set with its cardinality. Points 1 and 2 of Theorem 1.21 also continue to hold. For linear maps, we can still define them entirely by their action on the basis of a vector space. However, our connection to F^n has effectively disappeared, and matrices/coordinates aren't nearly as useful.

Where things really get interesting is with dual spaces and eigenvectors/eigenvalues. Let's get an example to start with. Continuing with the vector space we've been using this entire time, we can define the right shift operator R which acts in the following way

$$R(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots)$$

Suppose we're looking for an eigenvalue λ of R. Then we'd need for

$$\begin{aligned}(R - \lambda)(q_1, q_2, q_3, \dots) &= (0, 0, 0, \dots) \\ (0, q_1, q_2, \dots) - (\lambda q_1, \lambda q_2, \lambda q_3, \dots) &= (0, 0, 0, \dots)\end{aligned}$$

If $\lambda \neq 0$, then $q_1 = 0$, so $q_2 = 0$, and so on. If $\lambda = 0$, we get the same result, so $q_i = 0$. Thus, R has no eigenvalues or eigenvectors. This is not an isolated incident : linear operators in infinite-dimensional vector spaces are not guaranteed to have an eigenvalue or eigenvector. It follows that diagonalization and the Gram-Schmidt method are not guaranteed to work either. Similarly, there is no equivalent to the determinant or trace of an operator in infinite-dimensional vector spaces. The operator norm is still perfectly well-defined without any changes though!

Let's talk quickly about inner products as well. Inner-products are of course still perfectly well-defined in infinite dimensional vector spaces, no changes needed. Our definition of a hermitian operator is replaced with the more general self-adjoint operator.

Definition 4.5. *The adjoint $A^\dagger$ of a linear operator $A : V \to V$ is the unique operator $V \to V$ such that for any vectors $\vec{u}, \vec{v} \in V$, $\langle A^\dagger \vec{u}, \vec{v}\rangle = \langle \vec{u}, A\vec{v}\rangle$*

This turns out to be equivalent to the hermitian conjugate in finite-dimensional spaces. A self-adjoint operator is simply an operator that is its own adjoint.

Definition 4.6. *A unitary operator $A : V \to V$ is an operator such that $AA^\dagger = A^\dagger A = I$.*

Note that unlike the finite-dimensional case $AA^\dagger = I$ or $A^\dagger A = I$ on their own are not enough to guarantee that an operator is unitary (this has to do with some cardinality nonsense).

Finally, we talk about dual spaces. The definition of dual spaces remains perfectly valid, but the dual basis is no longer guaranteed to be a basis for the dual space (it is still called the dual basis though, just to be extra confusing). There's not much else to say, but I do want to go over one final interesting example.

$\mathbb{C}_c^\infty(\mathbb{R})$ is the set of all functions which are infinitely differentiable (i.e. can be differentiated an arbitrary number of times). In the dual space of this vector space lies the operator δ, which can be defined by the following for any $f(x) \in \mathbb{C}_c^\infty(\mathbb{R})$

$$\delta(f) = f(0)$$

You may recognize this, it's the Dirac delta function! There's a lot of other interesting stuff we can do with this definition, for example it allows us to extend the definition of a derivative to these dual operators. In fact, we can find that

$$\frac{d\delta}{dx}(f) = -\frac{df}{dx}(0)$$

5 Dirac Notation and QM

5.1 Introduction

Linear algebra lends itself to a plethora of applications in Physics, but one of the fields in Physics that benefits most from a mathematically formal understanding is quantum mechanics. Indeed, the formalism underlying quantum mechanics utilizes most if not all that has been covered in prior sections, and much more.

Note: Before proceeding, we note that the complex conjugate of $a \in \mathbb{C}$ will be referred to via a^* instead of $\overline{a}$ henceforth.

5.2 Defining a Hilbert space

Before proceeding with introducing quantum mechanics formalism, it is essential to first define a Hilbert Space, which the prior sections provide much of the machinery for. Some additional concepts from mathematical analysis (definition 5.1 - definition 5.4) need to be introduced prior to proceeding, which will briefly be summarized below.

Definition 5.1. *A metric space is simply a set, X, equipped with a metric $d : X \times X \to \mathbb{R}^+$ satisfying the following properties, for any $p, q, r \in X$:*

1. $d(p, q) = 0$ *iff* $p = q$
2. $d(p, q) = d(q, p)$
3. $d(p, r) \leq d(p, q) + d(q, r)$

The metric space is referred to as (X, d), or in cases where the metric d may be inferred or is well known, simply as X. Given X, it is easily possible to define a sequence that converges in X as follows:

Definition 5.2. *A sequence $\{p_n\}$ of elements in a metric space X converges in X to a point p if for all $\epsilon \in \mathbb{R}^+$, there exists a positive integer N, such that for $n \geq N$, $d(p_n, p) < \epsilon$.*

In practice, the sequence $\{p_n\}$ is usually termed a convergent sequence (in X is omitted), and sequences failing to satisfy this definition are referred to as divergent sequences. Convergent sequences can also be said to have a limit of p, i.e. that $\lim_{n\to\infty} p_n = p$. Another useful definition is that of a Cauchy sequence:

Definition 5.3. *A sequence $\{p_n\}$ of elements in a metric space X is said to be Cauchy if for all $\epsilon \in \mathbb{R}^+$, there exists a positive integer N, such that for $n, m \geq N$, $d(p_n, p_m) < \epsilon$.*

Cauchy sequences can be understood as formalizing the notion of elements in a sequence getting closer together. Intuitively, it would be ideal if convergent sequences are Cauchy and vice versa, but unfortunately the latter fails, that is, not all Cauchy sequences are convergent (in X). A key example of this is sequences in $\mathbb{Q}$ with the absolute value metric, wherein it is possible to construct sequences which converge to $\sqrt{2}$, but $\sqrt{2}$ is not itself in $\mathbb{Q}$. As a result, it would be beneficial to then classify spaces for which the notions of sequences being Cauchy and convergent are identical:

Definition 5.4. *A metric space X is said to be complete if every Cauchy sequence converges in X*

In general, working with complete metric spaces is rather nice, as the notion of elements getting closer to one another is consistent entirely with the notion of convergence and limit of a sequence. Through defining metric spaces and the notion of completeness, it is possible to define a Hilbert space, as follows:

Definition 5.5. *An inner product space, $(V, \langle \cdot, \cdot \rangle)$ is said to be a Hilbert space, if it is complete with respect to the metric d obtained from the norm prescribed by its inner product.*

To elaborate a bit more, all inner product spaces are normed spaces. To see this, one may easily construct the following norm, with the aid of the inner product:

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$$

It can easily be verified that this satisfies the properties of a norm. This prescribed norm also induces a metric on the inner product space, V, via:

$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$$

The notion of completeness in definition 5.5 refers to the completeness of the metric space (V, d). In general, this notion of completeness for vector spaces only requires a norm, and not an inner product. Normed vector spaces which are complete with respect to the metric obtained from the norm are called Banach Spaces. Hilbert spaces are special Banach spaces, in which the space with the natural norm prescribed by the inner product forms a Banach space - this provides them with additional structure. [10]

One may ask what the benefits of completeness are, apart from the fact that it aligns better with intuition for sequences, and that is quite a good question to ask at this stage. In general, it allows the avoiding of leaving the space in the limit, which the $\sqrt{2}$ example above demonstrates can easily happen. In what ways this can happen in quantum mechanics will be further elaborated on in the later sections, which, having defined a Hilbert space, can now be discussed.

[10] In finite dimensional vector spaces, all inner product spaces are also complete. This fails in infinite dimensional spaces, wherein the distinction becomes important.

5.3 Dirac (bra-ket) notation

To begin a discussion on quantum mechanics formalism, it is important to recognize that quantum mechanics is itself axiomatic - there are certain postulates which must be accepted or agreed upon to build a foundation. One such postulate is that the state of a system at a given instant is given by a ket, $|\psi\rangle$, in the space of states.

This, of course, may encourage a few questions - namely, what are kets, and the space of states? The space of states is a Hilbert space, with field $\mathbb{C}$, with its elements or vectors (notated $\vec{v}$ usually) notated via kets $|\psi\rangle$ instead. Since Hilbert spaces are vector spaces, $a\,|\psi\rangle$ is also a state for $a \in \mathbb{C}$. Another postulate in quantum mechanics is that the state $a\,|\psi\rangle$ represented by a multiple of another state $|\psi\rangle$ describes the same state - this is discussed in more detail past theorem 5.8. Then, any linear combinations of two kets produces another ket in the Hilbert space, and such linear combinations are understood to be a superposition of the states described by the two comprising kets. In most introductory cases, the Hilbert space considered in quantum mechanics is the $L^2(\mathbb{R})$ space with field $\mathbb{C}$, which loosely can be understood as a vector space of all square integrable functions on $\mathbb{R}$.[11]

To proceed further, we consider along with a Hilbert space, $\mathcal{H}$, its (continuous) dual space, $\mathcal{H}^*$, whose elements are referred to as co-vectors, forms/functionals on $\mathcal{H}$, or bras, $\langle\phi|$ (note that each functional/bra is continuous, by assumption). As for why considering the (continuous) dual space is natural, we want to be able to extract information from our states in a continuous way, which usually corresponds to outputting some value (usually real). This notation of using bras and kets is termed bra-ket notation, and its advantages will be seen throughout our upcoming discussion of quantum mechanics formalism.

With $\mathcal{H}$ and $\mathcal{H}^*$, the next motivation the reader may have is to enquire about the action of a linear form/functional, $\langle\phi|$, on a vector $|\psi\rangle$, compactly written as $\langle\phi|\psi\rangle \in F = \mathbb{C}$. For some intuition and concreteness, consider a finite n dimensional Hilbert space. Then a ket would be represented by a $n \times 1$ column vector, whereas a bra would then be represented by a $1 \times n$ row vector, so that the action of a bra on a ket, $\langle\phi|\psi\rangle$ corresponds to matrix multiplication, producing some $c \in \mathbb{C}$. While this is a good picture to have in mind conceptually, for the sake of providing some familiarity, it naturally breaks down for infinite dimensions and does not characterize the result of the action, c, in any notable way. This clearly suggests that our notion of action needs to be more general and informative.

Proceeding to generalize our notion of action of bras on kets requires better understanding the structures of $\mathcal{H}$ and $\mathcal{H}^*$. As already shown in theorem 3.17, given a basis $\mathcal{B}$ of a finite dimensional vector space V, $\mathcal{B}^*$ is a basis for V^*. Hence V and V^* have the same dimension, and are isomorphic (as Hilbert spaces are vector spaces, this is a more general result). Fortunately, this result holds for Hilbert spaces in general, if the continuous dual space is considered[12]. While isomorphisms indicate two objects are identical (bijective aspect of an isomorphism) wrt. the underlying structure imposed on them (with vector space structures, this corresponds to the linearity of the isomorphism),

[11] $L^2(X)$ space in actuality is a vector space of equivalence classes of square integrable functions on X, wherein functions are equivalent iff they differ on sets with measure 0, but it can help to treat it as a space of functions for now.

[12] The added structure of a Hilbert space $\mathcal{H}$ leads it to be isomorphic to its continuous dual space $\mathcal{H}^*$ in finite or infinite dimensions, via the Riesz representation theorem.

the finite dimensional case already demonstrates that representing a vector wrt. a basis and hence its dual functional wrt. the dual of said basis causes any isomorphism using the basis representation to depend on the chosen basis, via definition 3.16.

To elaborate on the above and see why it may be problematic, one may choose to use the bases $\mathcal{B}, \mathcal{B}^*$, while another may choose to use bases $\mathcal{B}_1, \mathcal{B}_1^*$, each of which produce different isomorphisms φ, φ_1 between $\mathcal{H}$ and $\mathcal{H}^*$. With these isomorphisms, $|\psi\rangle \in \mathcal{H}$ would generally correspond to different bras $\langle\phi|, \langle\phi_1| \in \mathcal{H}^*$, with the converse also holding true. Then, we remind the reader that a vector in any vector space as an object exists entirely independent of bases. To see this more concretely, $\vec{v} = (x, y) = (r, \theta)$ is the same vector in $\mathbb{R}^2$, and each bases simply allows for different representations of $\vec{v}$ that may be more convenient contextually. Similarly, $|\psi\rangle \in \mathcal{H}$, and $\langle\phi| \in \mathcal{H}^*$, as a state and a functional are independent from any choice of bases, and it would be desirable to have a natural correspondence, termed a canonical isomorphism, between a ket and bra, independent of this choice.

Indeed, the reader has already seen the machinery by which to construct this canonical isomorphism, in the example right below definition 3.14! Given $|\psi\rangle \in \mathcal{H}$, we can define a linear functional $\mathcal{H}^* \ni f_\psi : \mathcal{H} \to \mathbb{C}$, with $f_\psi(|\phi\rangle) = \langle|\psi\rangle, |\phi\rangle\rangle$, and consider the map $g : \mathcal{H} \to \mathcal{H}^*$, with $g(|\psi\rangle) = f_\psi$. To see that g is an isomorphism, let $f_\psi = 0_{\mathcal{H}^*}$. Then $f_\psi(|\phi\rangle) = \langle|\psi\rangle, |\phi\rangle\rangle = 0$. Since $|\phi\rangle$ is arbitrary, it follows that $|\psi\rangle = 0_{\mathcal{H}}$. Hence $\ker(g) = \{0_{\mathcal{H}}\}$, and g is injective. Surjectivity follows since each $|\psi\rangle$ maps to a different functional f_ψ, along with $\mathcal{H}$ and $\mathcal{H}^*$ being isomorphic. It can also be shown that g is anti-linear [13] (please bear with the notation for now - it will be simplified significantly soon):

$$\begin{aligned} g(a|\psi_1\rangle + b|\psi_2\rangle)(|\phi\rangle) &= f_{a\psi_1+b\psi_2}(|\phi\rangle) \\ &= \langle(a|\psi_1\rangle + b|\psi_2\rangle), |\phi\rangle\rangle \\ &= \langle|\phi\rangle, (a|\psi_1\rangle + b|\psi_2\rangle)\rangle^* \\ &= \langle|\phi\rangle, a|\psi_1\rangle\rangle^* + \langle|\phi\rangle, b|\psi_2\rangle\rangle^* \\ &= a^*\langle|\phi\rangle, |\psi_1\rangle\rangle^* + b^*\langle|\phi\rangle, |\psi_2\rangle\rangle^* \\ &= a^*\langle|\psi_1\rangle, |\phi\rangle\rangle + b^*\langle|\psi_2\rangle, |\phi\rangle\rangle = a^*g(|\psi_1\rangle)(|\phi\rangle) + b^*g(|\psi_2\rangle)(|\phi\rangle) \end{aligned}$$

Though g is anti-linear, we define g to be the canonical isomorphism between $\mathcal{H}$ and $\mathcal{H}^*$. One can however use g to construct an isomorphism, by defining $h : a|\phi\rangle \mapsto a^*|\phi\rangle$, with which $g \circ h$ becomes the desired isomorphism from $\mathcal{H}$ to $\mathcal{H}^*$. As all h does is permute the kets in $\mathcal{H}$, and g already provides us with anti-linearity, this is one of the reasons g itself is considered the isomorphism, and we now adopt the following notation, where $|\phi\rangle, |\psi\rangle \in \mathcal{H}$:

$$g(|\psi\rangle) = f_\psi = \langle\psi| \in \mathcal{H}^*$$

$$\langle\psi|\phi\rangle = f_\psi(|\phi\rangle) = \langle|\psi\rangle, |\phi\rangle\rangle$$

This syntactically justifies what initially seemed a rather strange choice of notation, and specifies the action of a bra on a ket. The relation

$$\langle\psi| \overset{\text{DC}}{\longleftrightarrow} |\psi\rangle$$

given by g is termed the dual correspondence (DC) between bras and kets, with the bra $\langle\psi|$ being the dual of $|\psi\rangle$ and vice versa. Another consequence of the above provides the dual of a linear

[13] Note that while g is anti-linear, each bra it generates is a linear functional. Essentially, only the correspondence between bras and kets is anti-linear, not the bras themselves.

combination of kets:

$$|\Phi\rangle = a\,|\phi\rangle + b\,|\psi\rangle \overset{\text{DC}}{\longleftrightarrow} a^*\,\langle\phi| + b^*\,\langle\psi| = \langle\Phi|$$

We can now discuss another reason for considering g to be the canonical isomorphism. It is clear that, for a given single ket, the definition of the inner product yields:

$$\langle\psi|\psi\rangle = f_\psi(|\psi\rangle) = \langle|\psi\rangle\,, |\psi\rangle\rangle = \||\psi\rangle\|^2$$

We hence need to verify that the above is consistent for linear combinations of kets as well. Taking $|\Phi\rangle\,, |\psi\rangle\,, |\phi\rangle$ as above, and using that each bra is a linear functional:

$$\begin{aligned}(a^*\,\langle\phi| + b^*\,\langle\psi|)(a\,|\phi\rangle + b\,|\psi\rangle) &= a^*\,\langle\phi|\,(a\,|\phi\rangle + b\,|\psi\rangle) + b^*\,\langle\psi|\,(a\,|\phi\rangle + b\,|\psi\rangle)\\ &= a^*a\,\langle\phi|\phi\rangle + a^*b\,\langle\phi|\psi\rangle + ab^*\,\langle\psi|\phi\rangle + b^*b\,\langle\psi|\psi\rangle\\ &= |a|^2\,\langle\phi|\phi\rangle + |b|^2\,\langle\psi|\psi\rangle + a^*b\,\langle\phi|\psi\rangle + ab^*\,\langle\phi|\psi\rangle^*\\ &= |a|^2\,\langle\phi|\phi\rangle + |b|^2\,\langle\psi|\psi\rangle + 2\,\mathrm{Re}\{a^*b\,\langle\phi|\psi\rangle\}\\ &= \langle\Phi|\Phi\rangle\end{aligned}$$

Thus, defining g as the canonical isomorphism and using the dual correspondence obtained from g lets the adopted notation provide a convenient representation of norms of kets in $\mathcal{H}$:

$$\langle\psi|\psi\rangle = f_\psi(|\psi\rangle) = \langle|\psi\rangle\,, |\psi\rangle\rangle = \||\psi\rangle\|^2,\ \forall\,|\psi\rangle \in \mathcal{H}$$

and we henceforth identify $\sqrt{\langle\psi|\psi\rangle}$ with the norm of any $|\psi\rangle \in \mathcal{H}$. Before proceeding further, we note below some useful results to assist the reader:

Theorem 5.6. $\langle\Phi|\,(a\,|\phi\rangle + b\,|\psi\rangle) = a\,\langle\Phi|\phi\rangle + b\,\langle\Phi|\psi\rangle$

Proof. This follows from the linearity of $\langle\Phi|$. □

Theorem 5.7. $(a\,\langle\phi| + b\,\langle\psi|)\,|\Phi\rangle = a\,\langle\phi|\Phi\rangle + b\,\langle\psi|\Phi\rangle$

Proof. This follows from the definition of vector addition of bras in the dual space. □

Theorem 5.8. $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$

Proof. Using the definition of the inner product: $\langle\phi|\psi\rangle = \langle|\phi\rangle\,, |\psi\rangle\rangle = \langle|\psi\rangle\,, |\phi\rangle\rangle^* = \langle\psi|\phi\rangle^*$ □

To provide some concreteness, consider going back to the finite n dimensional case with the use of bases. A ket $|\psi\rangle$ would be represented as a column vector $(v_1, \ldots, v_n)$ wrt the standard bases in $\mathbb{R}^n$. Then, a bra $\langle\psi|$ would be represented as a row vector $(v_1^*, \ldots, v_n^*)$ in the standard dual basis, so that the matrix product gives the inner product of $|\psi\rangle$ with itself (also its square norm) - which is not only consistent, but expands upon the picture we had initially!

To continue expanding our picture, we may now consider for each non-zero $|\psi\rangle \in \mathcal{H}$, the corresponding ket given by $|\Psi\rangle = \frac{1}{\||\psi\rangle\|}\,|\psi\rangle$. This corresponds to the same state as $|\psi\rangle$ by the second postulate of quantum mechanics. In particular, as $\||\psi\rangle\| \in \mathbb{R}^+$, its dual is $\langle\Psi| = \left(\frac{1}{\||\psi\rangle\|}\right)^*\langle\psi| = \frac{1}{\||\psi\rangle\|}\,\langle\psi|$, and via the theorems above, its norm is $\langle\Psi|\Psi\rangle = \frac{1}{\||\psi\rangle\|^2}\,\langle\psi|\psi\rangle = 1$!

Thus, every non-zero ket in a Hilbert space has associated with it a normalized ket, and since constant multiples of kets correspond to the same state, no generality is lost in considering any ket

in $\mathcal{H}$ apart from the zero ket to be normalized. Thus, we may adjust our definition for a quantum state to be a normalized ket $|\psi\rangle \in \mathcal{H}$, and this will be assumed henceforth unless otherwise stated.

In summary, we represent states and (continuous) functionals on these states using bra-ket notation, and we have been able to generalize and expand upon our intuition for vectors, their dual functionals and the actions of dual functionals on a vector, in finite dimensional Hilbert spaces to general ones through dual correspondence. We have also discussed the normalizability of non-zero kets, and hence associated each state with a normalized ket henceforth. The exercises above have already demonstrated some of the effectiveness of bra-ket notation: firstly that dual correspondence is essentially incorporated within it, and secondly that it simultaneously provides rather clean notation for norms of kets. The notation and formalism in introductory quantum mechanics usually seems awkward, clunky, or strange to most experiencing it for the first time, but we hope that the above provides some structure and motivation for bras, kets, and bra-ket notation as a whole.

5.4 Operators, Observables, and eigenstates

Before proceeding any further, one may ask why operators (definition 2.16) are essential in our discussion of quantum mechanics formalism, and the answer to this lies in another postulate of quantum mechanics: Every observable attribute of a physical system is described by linear operators acting on kets describing the state of the system. As such, observables such as position, momentum, energy and more, are all represented by operators acting on kets in state spaces. The proceeding sections thus tackle (linear) operators and their role in quantum mechanics.

5.4.1 Operators with bra-ket notation

With the aid of bra-ket notation, we may denote the action of an operator, T, on a ket, $|\psi\rangle$, as $T|\psi\rangle$. The action of an operator on a bra, $\langle\psi|T$ can be defined and understood to produce another bra in the dual space, which can act on a ket via $(\langle\psi|T)|\phi\rangle = \langle\psi|(T|\phi\rangle)$. With this definition, there is no ambiguity in the expression $\langle\psi|T|\phi\rangle$, as either notion of action are from definition equivalent. Operators can hence be thought of as acting from the right with bras, and acting from the left with kets.

Since $T|\psi\rangle$ is also a ket, a natural question to ask is what the dual of this ket is. The reader may be tempted to say that it is $\langle\psi|T$, but this is unfortunately not generally true. We define the dual of $T|\psi\rangle$ as follows:

$$T|\psi\rangle \overset{\text{DC}}{\longleftrightarrow} \langle\psi|T^\dagger$$

where $T^\dagger$ is defined to be the adjoint or Hermitian conjugate of the operator T, and is hence linear by definition. We call operators for which $T^\dagger = T$ Hermitian, which may sound familiar, as it is the precise definition we gave for the conjugate transpose of T. Indeed, if $\mathcal{H}$ is a finite dimensional Hilbert space, then $T^\dagger$ is simply the conjugate transpose of T, as the below will demonstrate formally:

Theorem 5.9. *if $\mathcal{H}$ has dimension n, where $n \in \mathbb{N}$, then $T^\dagger$ as defined above is the conjugate transpose of T*

Proof. As $\mathcal{H}$ is finite dimensional, $|\psi\rangle = \sum_{i=1}^n a_i e_i$, where $\mathcal{B} = \{e_i\}$ is a basis for $\mathcal{H}$. Then $\mathcal{B}^* = \{e_i^*\}$ is a dual basis for $\mathcal{H}^*$, and $\langle\psi| = \sum_{i=1}^n a_i^* e_i^*$. Let T_{ij} denote the entry in the i^{th} row and j^{th} column

of T (and similarly for $T^\dagger$). Then:

$$T\left|\psi\right\rangle = \begin{pmatrix} T_{11} & \dots & T_{1n} \\ \vdots & \ddots & \vdots \\ T_{n1} & \dots & T_{nn} \end{pmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n a_i T_{1i} \\ \vdots \\ \sum_{i=1}^n a_i T_{ni} \end{bmatrix}$$

and

$$\left\langle\psi\right| T^\dagger = \begin{bmatrix} a_1^* & \dots & a_n^* \end{bmatrix} \begin{pmatrix} T_{11}^\dagger & \dots & T_{1n}^\dagger \\ \vdots & \ddots & \vdots \\ T_{n1}^\dagger & \dots & T_{nn}^\dagger \end{pmatrix} = \begin{bmatrix} \sum_{j=1}^n a_j^* T_{j1}^\dagger & \dots & \sum_{j=1}^n a_j^* T_{jn}^\dagger \end{bmatrix}$$

Then, it follows that, since $\left\langle\psi\right| T^\dagger(T\left|\psi\right\rangle) = \langle T\psi|T\psi\rangle$ by definition of $T^\dagger$:

$$\begin{bmatrix} \sum_{j=1}^n a_j^* T_{j1}^\dagger & \dots & \sum_{j=1}^n a_j^* T_{jn}^\dagger \end{bmatrix} \begin{bmatrix} \sum_{i=1}^n a_i T_{1i} \\ \vdots \\ \sum_{i=1}^n a_i T_{ni} \end{bmatrix} = \sum_{k=1}^n \left(\sum_{i=1}^n a_i T_{ki} \right)^* \left(\sum_{i=1}^n a_i T_{ki} \right)$$

Then via matrix multiplication, it follows that:

$$\sum_{k=1}^n \left(\sum_{j=1}^n a_j^* T_{jk}^\dagger \right) \left(\sum_{i=1}^n a_i T_{ki} \right) = \sum_{k=1}^n \left(\sum_{i=1}^n a_i T_{ki} \right)^* \left(\sum_{i=1}^n a_i T_{ki} \right)$$

This implies that:

$$\left(\sum_{j=1}^n a_j^* T_{jk}^\dagger \right) = \left(\sum_{i=1}^n a_i T_{ki} \right)^*$$

Re-indexing $j = i$ then $k = j$, and using linearity and distributivity over multiplication of complex conjugation:

$$\left(\sum_{i=1}^n a_i^* T_{ij}^\dagger \right) = \left(\sum_{i=1}^n a_i^* T_{ji}^* \right)$$

Therefore $T_{ji}^\dagger = T_{ij}^*$, which implies that $T^\dagger$ is the conjugate transpose of T as required. □

Hence the newly suggested definition of an Hermitian operator agrees with the definition for finite dimensional Hilbert spaces, and generalizes well to infinite dimensions. Hermitian operators support the intuition we may have had with regards to dual correspondence, as the dual of $T\left|\psi\right\rangle$ is $\left\langle\psi\right| T^\dagger = \left\langle\psi\right| T$. The simplest examples of Hermitian operators are the zero operator and the identity operator. In fact, the identity operator being Hermitian is equivalent to dual correspondence:

$$I\left|\psi\right\rangle = \left|\psi\right\rangle \overset{\text{DC}}{\longleftrightarrow} \left\langle\psi\right| = \left\langle\psi\right| I$$

Motivated by this, we adopt the following notation for dual correspondence henceforth:

$$\left\langle\psi\right| = (\left|\psi\right\rangle)^\dagger,\ \left|\psi\right\rangle = (\left\langle\psi\right|)^\dagger$$

As an example of the utility of this notation, we prove the following

Theorem 5.10. $\langle\phi|T|\psi\rangle = \langle\psi|T^\dagger|\phi\rangle^*$

Proof. $\langle\phi|T|\psi\rangle = \langle\phi|\left(T\,|\psi\rangle\right)$. Using theorem 5.8 and the notation introduced above, $\langle\phi|\left(T\,|\psi\rangle\right) = \left(\left(T\,|\psi\rangle\right)^\dagger(\langle\phi|)^\dagger\right)^* = \left(\left(\langle\psi|\,T^\dagger\right)|\phi\rangle\right)^* = \langle\psi|T^\dagger|\phi\rangle^*$ □

As seen in the proof, the notation introduced makes it rather simple to refer to duals of kets, and of operators applied to kets (and the duals of their corresponding duals as well).

The reader may notice that the notion of an Hermitian operator is almost natural or intuitive, in the sense that it agreed with the naive guess we had for duals. This is not without reason, as will be explored in the next section.

5.4.2 Observables as Hermitian Operators:

Up until now, our entire discussion has mostly been centered around abstract mathematics. However, the goal of quantum mechanics, as well as most formalism in Physics, is to describe, model and characterize physical phenomenon. As such, a natural next step in our adventure through quantum mechanics formalism is to incorporate physical descriptions within our formalism, and to simplify these as much as possible.

Before proceeding, we prompt the reader to consider the special expression $\langle\psi|T|\psi\rangle$ - this is defined to be the expectation (or mean/average) value of the operator T in a given state $|\psi\rangle$. For an observable, $T = O$, we would like the expected or mean value in any quantum state to correspond to a real number (imaginary positions, energies and momenta do not carry physical significance after all). As a result, we would like the following equality, based on theorem 5.10, to hold:

$$c = \langle\phi|O|\phi\rangle \in \mathbb{R} \iff \langle\phi|O^\dagger|\phi\rangle = c^* = c = \langle\phi|O|\phi\rangle \tag{6}$$

Then, consider the following:

Theorem 5.11. *For $A, B \in \mathcal{L}(\mathcal{H})$, and $a, b \in \mathbb{C}$, $\langle\psi|(aA + bB)|\phi\rangle = a\langle\psi|A|\phi\rangle + b\langle\psi|B|\phi\rangle$*

Proof. Using the linearity of A, B and $\langle\psi|$:

$$\begin{aligned}\langle\psi|(aA + bB)|\phi\rangle &= \langle\psi|\left((aA + bB)|\phi\rangle\right)\\ &= \langle\psi|\left(aA|\phi\rangle + bB|\phi\rangle\right)\\ &= a\langle\psi|A|\phi\rangle + b\langle\psi|B|\phi\rangle\end{aligned}$$

which completes the proof. □

Using the above theorem with eq. (6), it follows that expectation values for O are real if and only if $0 = \langle\phi|O|\phi\rangle - \langle\phi|O^\dagger|\phi\rangle = \langle\phi|O - O^\dagger|\phi\rangle$. Since $|\phi\rangle$ is arbitrary, it follows that $O - O^\dagger = 0 \iff O = O^\dagger$ or in words, O is Hermitian. Essentially, for quantum mechanics to hold physical significance, we require Observables to be represented by Hermitian operators.

5.4.3 Determinate States and Eigenkets as Base Kets:

Next in quantum mechanics, we are interested in determinate states of an Observable, that is, states which are invariant under operation of an Observable. Since $a\,|\psi\rangle$, $a \in \mathbb{C}$ represents the same state as $|\psi\rangle$, invariance up to multiplication by a constant is sufficient. Specifically, we are interested in:

$$O\,|\psi\rangle = o\,|\psi\rangle\,,\ o \in \mathbb{C}$$

If this looks familiar, it should![14] This is indeed precisely referring to the eigenvalues and eigenvectors (eigenkets in this case) of the Observable, O. Our Observables being Hermitian operators leads to some nice properties, which we will state and provide brief proofs for below:

Theorem 5.12. *The eigenvalues of an Observable are real.*

Proof. To prove this, let $|\psi\rangle$ be an eigenstate of an Observable, O, and let $\langle\psi|\psi\rangle = a \neq 0$. Then the following holds:

$$\langle\psi|O|\psi\rangle = \langle\psi|\left(o\,|\psi\rangle\right) = o\,\langle\psi|\psi\rangle$$

Additionally, since $\langle\psi|\,O = (O\,|\psi\rangle)^\dagger = (o\,|\psi\rangle)^\dagger = o^*\,\langle\psi|$, by the same reasoning:

$$\langle\psi|O|\psi\rangle = \left(\langle\psi|\,O\right)|\psi\rangle) = o^*\,\langle\psi|\psi\rangle$$

Hence, it must hold that:

$$o\,\langle\psi|\psi\rangle = o^*\,\langle\psi|\psi\rangle \iff o = o^*$$

which uses the fact that eigenkets are not the 0 ket, which completes the proof. □

Theorem 5.13. *Eigenkets corresponding to distinct eigenvalues are orthogonal.*

Proof. Let o_1, o_2 be distinct eigenvalues of O, and $|\psi_1\rangle, |\psi_2\rangle$ the corresponding eigenkets. Then it follows that:

$$\langle\psi_1|O|\psi_2\rangle = o_2\,\langle\psi_1|\psi_2\rangle$$

Similarly, considering the action of O on $\langle\psi_1|$:

$$\langle\psi_1|O|\psi_2\rangle = o_1^*\,\langle\psi_1|\psi_2\rangle = o_1\,\langle\psi_1|\psi_2\rangle$$

where in the last equality, theorem 5.12 is used. Subtracting the 2 equations from each other yields:

$$0 = (o_2 - o_1)\,\langle\psi_1|\psi_2\rangle \iff \langle\psi_1|\psi_2\rangle = 0$$

via the distinctness of o_1 and o_2. Recalling $\langle\psi_1|\psi_2\rangle = \langle|\psi_1\rangle, |\psi_2\rangle\rangle$, this completes the proof. □

Another reason why eigenkets are so important can be seen almost immediately from prior work if we restrict to the finite dimensional case, by theorem 3.13. The entire Hilbert space $\mathcal{H}$ can be spanned by an orthonormal set of eigenkets for an observable O. With this in hand, it becomes possible to discuss the notion of measurement in quantum mechanics, and this is where the above result will showcase its utility.

5.5 Measurements in quantum mechanics

Based on our prior work, let O be an observable, let $|\psi_i\rangle$ be its eigenkets, and a_i be the (real-value) eigenvalues, with $1 \leq i \leq n$, where n is the dimension of $\mathcal{H}$. Then any state $|\phi\rangle$ may be expressed via $\sum_{i=1}^n c_i\,|\psi_i\rangle$. Using the fact that $|\phi\rangle$ is normalized, we may establish an important relation the coefficients c_i must satisfy, as below:

$$\begin{aligned}
1 = \langle\phi|\phi\rangle = \left(\sum_{j=1}^n c_j\,|\psi_j\rangle\right)^\dagger \sum_{i=1}^n c_i\,|\psi_i\rangle = \sum_{j=1}^n c_j^*\,\langle\psi_j|\sum_{i=1}^n c_i\,|\psi_i\rangle &= \sum_{i,j} c_j^* c_i\,\langle\psi_j|\psi_i\rangle \\
&= \sum_{i,j} c_j^* c_i \delta_{ij} \\
&= \sum_{i=1}^n |c_i|^2
\end{aligned}$$

[14] If not, don't worry, this material is hard!

In particular, this gives us that each $|c_i|^2 \leq 1$, and that the sum of the modulus of the coefficients must be 1. As for why this is important requires the introduction of another axiom of Quantum Mechanics - the Born rule.

The first statement of the Born rule is that the outcome of measuring a system in a state $|\phi\rangle$ for a quantity corresponding to the observable O is one of the eigenvalues a_j of O. In other words, the possible measurements of the quantity represented by observable O are quantized. This initially seems rather strange, but using the work above, we hope to make this somewhat more intuitive. The idea encapsulated by the Born rule is that the act of measurement changes the state - in other words, an observer interacting with the system causes the state of the system to change in the following way:

$$|\phi\rangle \to |\psi_j\rangle$$

Then, measuring the system for the quantity represented by the observable O would yield the expectation value $\langle\psi_j|O|\psi_j\rangle = \langle\psi_j|a_j|\psi_j\rangle = a_j \langle\psi_j|\psi_j\rangle = a_j$, which is the claim made by the Born rule. There is some ambiguity here however, as the system initially is in a superposition of states $|\psi_i\rangle$. What is it that decides which eigenket the system changes into? This is where the coefficients c_i, and the second statement of Born's rule come into play.

The second statement of Born's rule in our context is that each $|c_i|^2$ is a probability amplitude - that is, each $|c_i|^2$ describes the probability that the system changes from $|\phi\rangle$ to $|\psi_i\rangle$. Given what we previously derived, this is not far from reasonable, as each $|c_i|^2 \leq 1$, and the sum over all i equates exactly to 1. Rephrasing the above, $|c_i|^2$ gives the probability that an observer measures a_i for the quantity represented by O in the state $|\phi\rangle$. To give an example, if the system is in a state $|\phi\rangle = \frac{1}{\sqrt{2}}|\psi_1\rangle + \frac{1}{\sqrt{2}}|\psi_2\rangle$, then the observer is equally likely to measure a_1 or a_2 when measuring the system for the quantity represented by O. However, if the system is in the state $|\phi\rangle = \frac{\sqrt{3}}{2}|\psi_1\rangle + \frac{i}{2}|\psi_2\rangle$, then the probability of measuring a_1 is $\left|\frac{\sqrt{3}}{2}\right|^2 = \frac{3}{4}$, and the probability of measuring a_2 is $\left|\frac{i}{2}\right|^2 = \frac{1}{4}$, so an observer is more likely to measure a_1 than a_2 in this case.

The fact that quantum systems work in this way is rather bizarre, and many first seeing this would have a hard time believing that some of the postulates above lead to a physically meaningful theory. These postulates do, however, characterize quantum systems, and the next section provides a good example of such a system.

5.6 Stern Gerlach Experiment

We close off by providing an example of a quantum system. It is perhaps best to look at a system which inspired and led to the formulation of Quantum Mechanics itself - the Stern-Gerlach experiment. This involved firing collimated heated silver atoms through an inhomogeneous magnetic field, causing them to deflect, with the deflections recorded on a screen. This deflection would correspond to the z-component of the magnetic moment of the 47^{th} electron in the silver atom, and thus the z component of its spin (intrinsic) angular momentum. Classically, one would expect a spectrum on the screen, as angular momentum, and thus the magnetic moment of the electron would range through a continuum of values. Instead, the output on the screen was focused on only two points (fig. 1) in equal intensities. This led to the conclusion that electron spin angular momentum was indeed quantized - a clear departure from classical phenomenon.

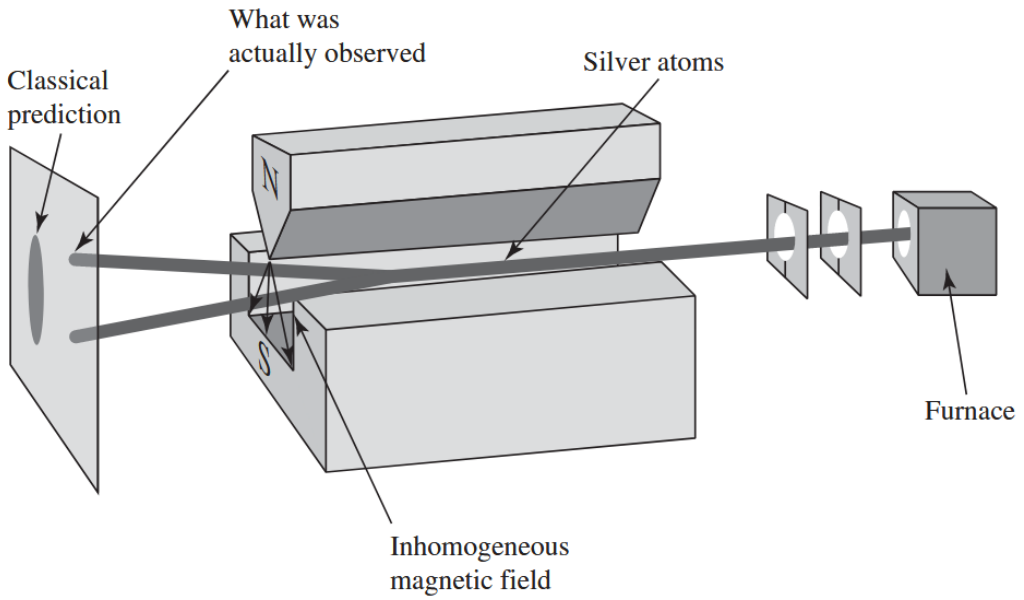


Figure 1: A figure detailing the set up of the Stern Gerlach ($\mathbf{S}_z$) apparatus. The sharpness of the north pole creates an inhomogenous magnetic field, which exerts a torque on the silver atoms, causing their deflection. Figure from Modern Quantum Mechanics: Third Edition by J.J. Sakurai and Jim Napolitano.

While the experiment was originally done in the z ($\mathbf{S}_z$ apparatus) direction, there is nothing particularly special about this choice, and the results may be replicated even in the x ($\mathbf{S}_x$ apparatus) and y ($\mathbf{S}_y$ apparatus) directions. Even more bizarre than the quantization result above, is the result of sequential Stern-Gerlach experiments. When one puts a collimated beam through an $\mathbf{S}_z$ apparatus, it filters into 2 parts, an $\mathbf{S}_z+$ and $\mathbf{S}_z-$ beam. Passing the $\mathbf{S}_z+$ beam through an $\mathbf{S}_x$ apparatus then further splits the beam into $\mathbf{S}_x+$ and $\mathbf{S}_x-$ beams. This suggests, at least intuitively, that the $\mathbf{S}_z+$ beam is comprised in 1:1 ratio of atoms with $\mathbf{S}_x+$ and $\mathbf{S}_x-$ components in spin angular momentum. In particular, selecting the $\mathbf{S}_x+$ beam from above, we expect it to comprise of atoms with $\mathbf{S}_x+$ and $\mathbf{S}_z+$ components. This immediately breaks down however, as putting this beam through the $\mathbf{S}_z$ apparatus splits it again into $\mathbf{S}_z+$ and $\mathbf{S}_z-$ beams in equal proportions. As for why this happens, it turns out that measuring $\mathbf{S}_z$ and $\mathbf{S}_x$ simultaneously is not possible, meaning that when the $\mathbf{S}_z+$ beam was passed through the $\mathbf{S}_x$ apparatus, all information about the system was lost. In the language of operators, this specifically means that the operators corresponding to $\mathbf{S}_x$ and $\mathbf{S}_z$ do not commute.

5.7 Introducing the Commutator

Definition 5.14. *Given 2 operators, A and B, we define the commutator:*

$$[A, B] = AB - BA$$

For operators that do commute, the commutator returns the 0 or null operator. It is fairly clear to see that any operator commutes with itself. For a more concrete example, consider the energy and momentum of a free particle (no potential). If the particle has momentum p, then the particle has energy (entirely kinetic) $\frac{p^2}{2m}$, and the operators mimic this behavior. As a result, knowing the

energy in fact provides you with the momentum of the particle, and vice versa. Therefore:

$$[p, H_f] = 0$$

where H_f is the Hamiltonian, or energy operator of a free particle. As an example of operators that do not commute, it is in fact a postulate that the position and momentum operators satisfy:

$$[x, p] = xp - px = i\hbar$$

That is, the operators do not commute. In other words, it is not possible to know both the position and momentum of a system precisely, which is in fact the famous Heisenberg uncertainty relation. For another example that is not just a postulate, we may relate back to our Stern Gerlach experiment discussion from before. What the sequential experiments show is in fact:

$$[\mathbf{S}_z, \mathbf{S}_x] \neq 0$$

and this is precisely why information about the particle momenta is lost after the beam passes through the $\mathbf{S}_x$ apparatus, much like with measuring position and momentum. The commutator satisfies several properties, some of which have been stated and proven below:

Proposition 5.1. *For operators A, B, C, the commutator satisfies the following properties:*

- $[A, B] + [B, A] = 0$
- $[A + B, C] = [A, C] + [B, C]$
- $[A, B + C] = [A, B] + [A, C]$
- $[A, BC] = [A, B]C + B[A, C]$
- $[AB, C] = [A, C]B + A[B, C]$

Proof. The proof of each property is given in order below:

- $[A, B] = AB - BA = -(-AB + BA) = -(BA - AB) = -[B, A]$
- $[A + B, C] = (A + B)C - C(A + B) = (AC - CA) + (BC - CB) = [A, C] + [B, C]$
- Identical to the proof right above.
- $[A, BC] = ABC - BCA = ABC - BAC + BAC - BCA = (AB - BA)C + B(AC - CA) = [A, B]C + B[A, C]$
- Identical to the proof right above.

□

While a lot more can be said about Quantum Mechanics, we have at the very least scratched the surface of its underlying formalism, and we hope the reader found this to be a digestible and useful introduction to Quantum Mechanics, along with enjoying the application of some of the Linear Algebra learnt prior!

6 Angular Momentum in QM

6.1 Spin Operators

For the entirety of this introduction we have been concerned with the mathematical foundations necessary for a complete understanding of linear algebra and its formal applications with quantum mechanics. Yet, as we have talked about these behaviors, we have not given a physical application of the formalism that we have so carefully set up. As such, we shall use this final section to introduce one of the most well-known and arguably most important application of linear algebra in quantum mechanics.

Spin is an inherent property of particles as well as other objects that behave quantum mechanically, such as atoms. Contrary to its naming convention, spin is not the indication that the particle itself is rotating, but rather an independent physical property that has been experimentally verified via the Stern-Gerlach experiment. This name "spin" instead comes from the fact that the property itself behaves very similarly to the physical property of angular momentum.

As we have previously seen, an operator in quantum mechanics require us to define two pieces of information, the Hilbert space of the operator and its commutation relations. Let us begin by writing out the commutation relations. Before we jump into spin, however, let us start by looking at the so called *canonical commutation relation.*

Definition 6.1. *Let r_i denote the position operator of the i-th direction and p_j denote the momentum operator of the j-th direction. The commutation relation between x_i and p_j is defined to be as follows:*

$$[r_i, p_j] = i\hbar\delta_{ij}$$

where δ_{ij} is the Kronecker delta. This is known as the canonical commutation relation.

The derivation of this commutation relation is often treated in graduate level quantum mechanics courses and the implications of this commutation relation is often treated in any undergraduate level quantum mechanics course. For now, we shall simple treat this relation as an axiom for the remainder of the material. However, now that we have the commutation relation for the position and momentum operators, we can derive the commutation relation of the angular momentum operators.

Recall from classical mechanics that angular momentum is defined as

$$\vec{L} = \vec{r} \times \vec{p}$$

or rather

$$L_i = \varepsilon_{ijk} r_j p_k$$

where ε_{ijk} is the Levi-Civita symbol. Since we have the canonical commutation relation, we can then derive the commutation relations between the components of angular momentum. We can perform this explicitly.

Example 6.2. *Consider the angular momentum operator in the x-direction, L_x, and the angular momentum operator in the y-direction, L_y. We can explicitly write the operators using the definition of angular momentum from classical mechanics.*

$$L_x = r_y p_z - p_y r_z$$
$$L_y = r_z p_x - p_z r_x$$

Thus, their commutation relation follows as

$$\begin{aligned}
[L_x, L_y] &= [r_y p_z - p_y r_z, r_z p_x - p_z r_x] \\
&= [r_y p_z, r_z p_x] - [r_y p_z, p_z r_x] - [p_y r_z, r_z p_x] + [p_y r_z, p_z r_x] \\
&= r_y [p_z, r_z] p_x + [r_y, r_z] p_z p_x + r_z r_y [p_z, p_x] + r_z [r_y, p_x] p_z \\
&\quad - r_y [p_z, p_z] r_x - [r_y, p_z] p_z r_x - p_z r_y [p_z, r_x] - p_z [r_y, r_x] p_z \\
&\quad - p_y [r_z, r_z] p_x - [p_y, r_z] r_z p_x - r_z p_y [r_z, p_x] - r_z [p_y, p_x] r_z \\
&\quad + p_y [r_z, p_z] r_x + [p_y, p_z] r_z r_x + p_z p_y [r_z, r_x] + p_z [p_y, r_x] r_z \\
&= -i\hbar \left(r_y p_x - p_y r_x \right) \\
&= i\hbar \left(p_y r_x - r_y p_x \right)
\end{aligned}$$

Now, recall that the angular momentum operator in the z-direction is given by

$$L_z = r_x p_y - p_x r_y = p_y r_x - r_y p_x$$

and thus, we arrive at

$$[L_x, L_y] = i\hbar L_z$$

as the commutation relation.

The remaining commutation relations are left as an exercise although their construction is nearly identical to the above. We find that

$$[L_x, L_z] = -i\hbar L_y$$

$$[L_y, L_z] = i\hbar L_x$$

which can all be summarised as

$$[L_i, L_j] = i\hbar \varepsilon_{ijk} L_k$$

In addition to the commutation relation between components of the angular momentum, we may also find the commutation relation between any component of the angular momentum and the square of the magnitude of the angular momentum. Recall that the square of the magnitude is found to be

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

It then follows that the commutation relation is derived as

$$\begin{aligned}
[L^2, L_z] &= [L_x^2 + L_y^2 + L_z^2, L_z] \\
&= [L_x^2, L_z] + [L_y^2, L_z] + [L_z^2, L_z] \\
&= L_x [L_x, L_z] + [L_x, L_z] L_x + L_y [L_y, L_z] + [L_y, L_z] L_y + L_z [L_z, L_z] + [L_z, L_z] L_z \\
&= -i\hbar L_x L_y - i\hbar L_y L_x + i\hbar L_y L_x + i\hbar L_x L_y \\
&= 0
\end{aligned}$$

While this commutation relation is derived for L^2 with L_z, we can easily see that the commutation relation holds for L_x and L_y, the calculation of which is also left as an exercise.

As we stated previously, spin is an inherent property that is analogous to angular momentum. Thus, we find that the commutation relations for the spin operators are identical to that of the angular momentum operators. We are left with the following six key commutation relations

$$[S_x, S_y] = i\hbar S_z, \qquad [S_y, S_z] = i\hbar S_x, \qquad [S_z, S_x] = i\hbar S_y$$

$$[S^2, S_x] = 0, \qquad [S^2, S_y] = 0, \qquad [S^2, S_z] = 0$$

which define the spin operators.

Note: We can also define the commutation relations with other operators, such as the position, momentum, and angular momentum operators, but all these commutators are trivial.

Now that we have our commutation relations defined, we can continue to define the Hilbert space of the spin operators. To do so, we have two things to do. First, we should pick good quantum numbers. Recall that good quantum numbers correspond to operators that commute with each other. Thus, we can choose our quantum numbers to correspond to S^2 and any component of the spin. The conventional choice is S_z. Secondly, we can again refer to angular momentum to give us insight into how our spin Hilbert space should be defined.

Let us once again return to angular momentum then. It is too much for us to derive here, as doing so requires knowledge in both multivariable calculus and, more importantly, partial differential equations. Regardless, if one would try to solve the Schrödinger equation in spherical coordinates, they would find that the eigenvalues of L^2 and L_z are

$$L^2 |\psi\rangle = \hbar^2 l(l+1) |\psi\rangle$$

$$L_z |\psi\rangle = \hbar m_l |\psi\rangle$$

where l is the azimuthal quantum number and m_l is the magnetic quantum number or the angular momentum projection. It is also found that l and m_l are bounded such that $l \in \mathbb{Z}_{\geq 0}$, and $m_l \in \mathbb{Z}$ with $-l \leq m_l \leq l$. This then allows us to write any angular state as a unique combination of l and m_l.

Note: There is a method to derive these operators without referencing techniques of partial differential equations which will be revisited in the subsection 6.2 once we have an understanding of ladder operators.

There is, however, one problem with bringing these quantum numbers into our spin as is. As we have established, spin is an intrinsic property of a particle. Because of this, it would make sense that all particles of a particular type has the same overall spin s, which is analogous to our azimuthal quantum number l. They may differ on their spin projection m_s, which is analogous to our magnetic quantum number m_l. Thus, there are a total of $2s - 1$ possible spin states for any given particle of spin s.

The problem appears when we consider the Stern-Gerlach experiment. The experiment shows us that particles passing through an external magnetic field will discretely split based on their spin state. Furthermore, the experiment showed us that there were two spin states for the used particles. Yet, if the quantum number s is bound to only integer values, then there will always be an odd number of possible spin states. Therefore, the spin quantum number s must be a non-negative half-integer value.

$$s \in \frac{1}{2}\mathbb{Z}_{\geq 0}$$

The spin projection quantum number is still restricted to $-s \leq m_s \leq s$ and increases/decreases in integer values.

To sum this up, we can write any spin state using two quantum numbers, denoted s and m_s, such that the eigenstates have the following eigenvalues.

$$S^2 |s, m_s\rangle = s(s+1)\hbar^2 |s, m_s\rangle$$

$$S_z |s, m_s\rangle = m_s \hbar |s, m_s\rangle$$

Furthermore, our four operators follow the commutation relations shown below.

$$[S_x, S_y] = i\hbar S_z, \qquad [S_y, S_z] = i\hbar S_x, \qquad [S_z, S_x] = i\hbar S_y$$

$$[S^2, S_x] = 0, \qquad [S^2, S_y] = 0, \qquad [S^2, S_z] = 0$$

Note: For the following three final sections, we will be treating $\hbar = 1$ as writing the additional terms is tedious. Furthermore, the notation is consider standard and should a reader need to recall where the $\hbar$ is located, they are recommended to look back at the previous sections.

6.2 Ladder Operators

While we have made some headway into understanding this intrinsic spin, there is still quite a bit of ground that can be covered here. For example, while we have know how S^2 and S_z operate on our states, since they are eigenstates, we do not know how S_x and S_y operates on the same states. Of course, one could move basis such that the quantum numbers used correspond to S_x and S^2 instead of S_z and S^2 but doing so is tedious, requires us to know how to move to said basis, and results in the same issue with S_x now replaced for S_z. However, we can use what we have already established to find how these other two operators operate in this basis. We first move directly into this secret tool, known as a ladder operator.

Definition 6.3. *Consider some operator N and its eigenstate $|n\rangle$ with eigenvalue n. Suppose that there is an operator X and constant $c \in \mathbb{C}$ such that*

$$[N, X] = cX$$

Then, the operator NX applied onto the state $|n\rangle$ gives

$$\begin{aligned} NX |n\rangle &= XN |n\rangle + [N, X] |n\rangle \\ &= Xn |n\rangle + cX |n\rangle \\ &= nX |n\rangle + cX |n\rangle \\ &= (n+c)X |n\rangle \end{aligned}$$

Thus, either $X |n\rangle = 0$ or the produced state is an eigenstate of N with an eigenvalue of $n + c$ (i.e. $X |n\rangle = |n + c\rangle$). We thus define X as a ladder operator for N. It is known as a raising ladder operator if c is real and positive and a lowering operator if c is real and negative. By properties of the adjoint, if N is a Hermitian operator, it can be shown that

$$\left[N, X^\dagger\right] = -cX^\dagger$$

and thus, if X is a raising operator, then $X^\dagger$ is a lowering operator and vice versa.

For the simplest and most applicable uses of these ladder operators, let us define S_+ to be a raising ladder operator for S_z with $c = +1$ and S_- to be a lowering ladder operator for S_z with $c = -1$. It should be further noted that S_+ and S_- commute with S^2, as they only raise and lower the projection quantum number m_s and not the spin quantum number s. That is to say that it does not matter which order the operators S_+ and S^2 are applied in, as either was we receive the same result.

$$S_+S^2 \left|s,m\right\rangle = S^2S_+ \left|s,m\right\rangle = s(s+1)a \left|s,m+1\right\rangle$$
$$S_-S^2 \left|s,m\right\rangle = S^2S_- \left|s,m\right\rangle = s(s+1)a \left|s,m-1\right\rangle$$

There is one more relation the connects the above operators back to our original S_z

$$[S_+, S_-] = 2S_z$$

which we may find useful in the coming portions.

It stands to reason that S_+ and S_- may also have some relation to S_x and S_y, although currently the relationship is unclear. In order to derive what this relation may be, we must presume what form these two operators must take. To begin, we can assume that our operator only involves S_x and S_y and no other operators. Since the operators only increase or decrease the projection of the spin by one quanta, it makes sense that they should have the same units as our operators S_z, S_y, and S_x. In other words, S_+ and S_- should be linear in S_x and S_y.

$$S_+ = \alpha S_x + \beta S_y$$

$$S_- = \gamma S_x + \eta S_y$$

where $\alpha, \beta, \gamma, \eta \in \mathbb{C}$. With four unknowns here, we of course need at least four equations to find S_+ and S_-.

We can begin with the simple statement that $S_+^\dagger = S_-$. $S_+^\dagger$ can be written using the form above as

$$S_+^\dagger = \alpha^* S_x^\dagger + \beta^* S_y^\dagger = \alpha^* S_x + \beta^* S_y$$

since S_x and S_y are Hermitian observables. Thus, our first two equations are

$$\alpha^* = \gamma, \qquad \beta^* = \eta$$

Then, we can turn to our commutation relations. Our commutation relations with S_z gives us the following:

$$[S_z, S_+] = [S_z, \alpha S_x + \beta S_y] = \alpha \left[S_z, S_x\right] + \beta \left[S_z, S_y\right] = i\alpha S_y - i\beta S_x = S_+ = \alpha S_x + \beta S_y$$

or rather

$$\alpha = -i\beta, \qquad \beta = i\alpha$$

which are equivalent statements. We can note that the respective commutation relation with S_- reveals similar information, which unfortunately is again redundant.

$$[S_z, S_-] = [S_z, \gamma S_x + \eta S_y] = \gamma \left[S_z, S_x\right] + \eta \left[S_z, S_y\right] = i\gamma S_y - i\eta S_x = -S_- = -\gamma S_x - \eta S_y$$

$$\gamma = i\eta, \qquad \eta = -i\gamma$$

We do have one last commutation relation, however, that we can use. Applying what we have previously suggested that our operators look like, we have

$$[S_+, S_-] = [\alpha S_x + \beta S_y, \gamma S_x + \eta S_y] = \alpha\gamma\,[S_x, S_x] + \beta\gamma\,[S_y, S_x] + \alpha\eta\,[S_x, S_y] + \beta\eta\,[S_x, S_y] = 2S_z$$

$$(i\alpha\eta - i\beta\gamma)\,S_z = 2S_z$$

Combining everything gives us

$$(i\alpha\,(-i\alpha^*) - i\,(i\alpha)\,\alpha^*) = 2\,|\alpha|^2 = 2$$

which then finally gives us that $|\alpha| = 1$. There still is some ambiguity with the phase of α, which in turn will define the phase of the remainder of the coefficients. However, this phase is not physically significant and, thus, we have freedom to pick any phase that we wish. We will then take the conventional choice which gives $\alpha = 1$. We then find that our ladder operators are written as follows

$$S_+ = S_x + iS_y$$
$$S_- = S_x - iS_y$$

Let us then finalize our discussion of the ladder operators by seeing how the are applied upon our S_z basis states. From our definition of ladder operators in Definition 6.3, we know that the application of the ladder operators upon an S_z eigenstates gives an S_z eigenstate in which the projection quantum number has been increased or decreased by one quanta. We then simply know

$$S_+\,|s, m\rangle = \alpha\,|s, m+1\rangle$$

$$S_-\,|s, m\rangle = \beta\,|s, m-1\rangle$$

where α and β are unknown scalars such that $\alpha, \beta \in \mathbb{C}$. We can use our commutation relations and definition of S_+ and S_- to find what these scalars may be. We will begin by finding the expectation value of S_+S_-.

Example 6.4. *Consider the raising and lowering operators for S_z, S_+ and S_-, as well as some arbitrary eigenstate of S_z, $|s, m\rangle$. The expectation value of S_+S_- can be found as*

$$\begin{aligned}\langle s, m\,|S_+S_-|\,s, m\rangle &= \left\langle s, m\,\middle|\,S_-^\dagger S_-\,\right|\,s, m\right\rangle \\ &= \langle s, m\,|\beta^*\beta|\,s, m\rangle \\ &= |\beta|^2\,\langle s, m|s, m\rangle = |\beta|^2\end{aligned}$$

We can then write S_+S_- in terms of S^2 and S_z. Using our expansion of S_+ and S_-, we have the following

$$\begin{aligned}S_+S_- &= (S_x + iS_y)\,(S_x - iS_y) \\ &= S_x^2 + S_y^2 + iS_yS_x - iS_xS_y \\ &= S_x^2 + S_y^2 - i\,[S_x, S_y] \\ &= S^2 - S_z^2 + \hbar S_z\end{aligned}$$

where $\hbar$ has been included in order to make the units match between terms. Applying the above to our eigenstate $|s, m\rangle$ gives us

$$S_+S_-\,|s, m\rangle = (S^2 - S_z^2 + \hbar S_z)\,|s, m\rangle = \hbar^2\left(s\,(s+1) - m^2 + m\right)|s, m\rangle$$

which can be rewritten as $\hbar^2 (s+m)(s-m+1)$*. Once again, we are left with an arbitrary phase for this scalar coefficient* β*, but once again this phase are not physically significant so we can choose our scalar to be real and positive. Therefore, we are left with*

$$\beta = \hbar\sqrt{(s+m)(s-m+1)}$$

Likewise, the coefficient α is found to be

$$\alpha = \hbar\sqrt{(s-m)(s+m+1)}$$

It is important to notice that these coefficients have also ingrained within them the fact that m cannot be greater than s and cannot be less than $-s$. That is to say, the following is given:

$$S_+ |s, s\rangle = 0$$

$$S_- |s, -s\rangle = 0$$

and we now have a complete pictures of our ladder operators.

We can then define our operators S_x and S_y by referencing the expressions we had found earlier. We find that

$$S_x = \frac{1}{2}(S_+ + S_-)$$

$$S_y = \frac{1}{2i}(S_+ - S_-)$$

which fully define their operation onto the S_z eigenstates. Let us revisit this in a moment.

We had previously promised to revisit the possible values for the spin quantum number and projection quantum number. Let's do this now.

Example 6.5. *Let us begin by considering some arbitrary eigenstate* $|A, B\rangle$*, where* A *is the eigenvalue of* S^2 *and* B *is the eigenvalue of* S_z*. We can rewrite* S^2 *as*

$$S^2 = S_x^2 + S_y^2 + S_z^2$$

or rather

$$S_x^2 + S_y^2 = S^2 - S_z^2$$

This operator can then be applied onto our eigenstate.

$$(S_x^2 + S_y^2)|A, B\rangle = (S^2 - S_z^2)|A, B\rangle$$

Now, since S_x *and* S_y *are observables, their eigenvalues are real. Thus, the left hand side of our equation here must have a positive coefficient, or rather that* $S^2 - S_z^2$ *must be nonnegative. More importantly, we must have that* $|B| \leq \sqrt{A}$*, establishing an upper and lower boundary for* B*. Let us then turn our attention to other operators. Consider the raising and lowering operators* S_+ *and* S_- *for* S_z*. Remember that* S_+ *and* S_- *are defined such that they increase and decrease the* S_z *eigenvalue* B *by one quanta, respectively. However, we now also know that* B *is bounded above and below by the value of* A*. Thus, let us suppose that this upper value is* B_+ *and this lower value is* B_-*. Since these are our upper and lower bounds, we have no state whose* S_z *eigenvalue is higher than* B_+ *and no state whose* S_z *eigenvalue is lower than* B_-*. Applying the the raising and lowering operators onto these states gives us*

$$S_+ |A, B_+\rangle = 0$$

$$S_- |A, B_-\rangle = 0$$

Furthermore, if we apply the raising and lowering operators to the other states, we get

$$S_- S_+ |A, B_+\rangle = 0$$

$$S_+ S_- |A, B_-\rangle = 0$$

Now, as we saw in Example 6.4, we can rewrite these operators in terms of S^2 and S_z. We find that

$$S_- S_+ |A, B_+\rangle = (S^2 - S_z^2 - S_z) |A, B_+\rangle = A - B_+^2 - B_+ = 0$$
$$S_+ S_- |A, B_-\rangle = (S^2 - S_z^2 + S_z) |A, B_-\rangle = A - B_-^2 + B_- = 0$$

which can then be combined via subtraction to find the following condition on B_+ and B_-

$$(B_+ + B_-)(B_- - B_+ - 1) = 0$$

As we have defined, $B_+ > B_-$, so the second term must be negative. Thus, we have that $B_+ + B_- = 0$ or rather that $B_- = -B_+$. This gives us the sum of the two numbers, but we can also look at the difference between B_+ and B_-. Recall that the spin is quantized and, thus, in order for us to move from B_- to B_+, the raising operator must be applied an integer number of times. Thus, the difference is an integer value, or rather

$$B_+ - B_- = 2s$$

where $s \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. It subsequently follows from the fact that $B_- = -B_+$ that

$$B_+ = s, \qquad B_- = -s$$

and the allowable values of B spans all the integer values between, and including, the two above. We can then apply $B_+ = s$ into the equation above to find the eigenvalue of S^2

$$A - s^2 - s = 0 \quad \Rightarrow \quad A = s^2 + s = s(s+1)$$

We can pick s to be the quantum number for S^2 and define m to be the quantum number for S_z such that

$$S^2 |s, m\rangle = s(s+1) |s, m\rangle \qquad \text{for } s \in \frac{1}{2}\mathbb{Z}_{\geq 0}$$
$$S_z |s, m\rangle = m |s, m\rangle \qquad \text{for } -s \leq m \leq s \text{ and } (m_i - m_j) \in \mathbb{Z}$$

Before we finish this section, let us bring up something rather concrete. As we have somewhat geared up to in this entire formulation, we can represent our operators in terms of matrices. The general form of these matrices is a bit extreme to do in such a short amount of time. Thus, we shall focus on one of the most well-known examples of these matrices, the Pauli matrices. These matrices represent these operators in a spin-$\frac{1}{2}$ system. We can find these matrices by analyzing the behavior of the operators on our states. To begin with, however, we need a representative of our states.

The most common method of representing these states is an object known as a spinor. A spinor is a pseudovector or an axial vector, and so remember that while a spinor may look like a vector, it does

not truly represent one. For spin-$\frac{1}{2}$ systems, we only have two states, which we often colloquially call spin-up ($|1\rangle$) and spin-down ($|0\rangle$). As spinors, we typically write these to be

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Now, we defined these states to be eigenstates of S_z and S^2, meaning that in this above representation, we can simply write S_z and S^2 as diagonal matrices where the diagonal entries are the respective eigenvectors

$$\begin{aligned}
S_z = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad &\rightarrow \quad S_z|1\rangle = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}|1\rangle \\
&\rightarrow \quad S_z|0\rangle = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2}|0\rangle \\
S^2 = \frac{1}{2}\left(\frac{1}{2}+1\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad &\rightarrow \quad S^2|1\rangle = \frac{1}{2}\left(\frac{1}{2}+1\right)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\left(\frac{1}{2}+1\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\left(\frac{1}{2}+1\right)|1\rangle \\
&\rightarrow \quad S^2|0\rangle = \frac{1}{2}\left(\frac{1}{2}+1\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\left(\frac{1}{2}+1\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\left(\frac{1}{2}+1\right)|0\rangle
\end{aligned}$$

Repeating the same for the remaining four operators is slightly more difficult. We can begin by looking at the raising and lowering operators. We know that $S_+|1\rangle = 0$ and $S_-|0\rangle = 0$ as these states are, respectively, the largest and smallest quantities of the projection. Thus, they cannot be raised or lowered, respectively. The remaining relation is given by our known application of S_+ or S_- onto some arbitrary state $|s, m\rangle$

$$\begin{aligned}
S_+|1\rangle = 0, \quad S_+|0\rangle = \sqrt{\left(\frac{1}{2}+\frac{1}{2}\right)\left(\frac{1}{2}-\frac{1}{2}+1\right)}|1\rangle \quad &\Rightarrow \quad S_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
S_-|1\rangle = \sqrt{\left(\frac{1}{2}+\frac{1}{2}\right)\left(\frac{1}{2}-\frac{1}{2}+1\right)}|0\rangle, \quad S_-|0\rangle = 0 \quad &\Rightarrow \quad S_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}
\end{aligned}$$

Finally, by using the knowledge that $\frac{1}{i} = -i$, we can the use the raising and lowering operators to derive the matrices for S_x and S_y in this representation. We find the following

$$S_x = \frac{1}{2}(S_+ + S_-) = \frac{1}{2}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$S_y = \frac{-i}{2}(S_+ - S_-) = \frac{-i}{2}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

The matrices are given here are the conventional forms of the spin-$\frac{1}{2}$ matrices. They are directly related to the so called Pauli matrices σ_j through the following:

$$S_x = \frac{1}{2}\sigma_1, \quad S_y = \frac{1}{2}\sigma_2, \quad S_z = \frac{1}{2}\sigma_3$$

and in some cases, the Pauli matrices are used instead of S_x, S_y, and S_z when dealing specifically with spin-$\frac{1}{2}$ systems.

Note: We could have easily taken a different representation. For example, we could have stated that

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

in which case our matrices would change. However, the commutation relations, eigenvalues, eigenstates, and so on are not defined by the representation of the operator. As a result, these properties would persist.

6.3 Adding Spin

As we bring this session to a close, we want to note one rather important thing. In the previous two sections, we had constructed two very important parts for spin. The first is the creation of the spin operators and Hilbert space. In that section, we defined S_x, S_y, S_z, and S^2 as well as defined two "good" quantum numbers to define our states, which we called s and m_s. In the second section, we discussed the aptly named ladder operators S_+ and S_- which, respectively, can increase or decrease the projection quantum number m_s of a state. We defined their commutation relations and showed their connection to the operators S_x and S_y and even demonstrated that they cannot create an disallowed state.

However, in this entire discussion, there has been one thing that is missing. We have a way for us to move between different values of the projection quantum number, but do not have a way to move between different total spin values. Of course, we need to be careful when we consider this. For any given particle, the total spin value s is a constant of nature. This sole fact makes it seem strange for there to be such an operator that increases or decreases the total spin quantum number. This is benefited by the fact that such an operator cannot exist. As stated above in Definition 6.3, a raising ladder operator for and operator A is one that increases the eigenvalue of A by some set scalar value c. However, the spacing between the eigenvalues of S^2 is not constant as the eigenvalues are quadratic in s.

Yet, while one may not be able to change the total spin quantum number of a single particle, a collection of particles may not have the same total spin quantum number as any single particle in the collection. After all, when we combine two angular momenta in classical mechanics, we do not expect the total angular momentum to be the same. However, these values do not add simply in quantum mechanics, even when it comes to our spin values. Unfortunately, the intricacies of angular momentum/spin addition in quantum mechanics is not found in linear algebra, but rather can be given from group theory. As such we will not be discussing these intricacies here.

That, however, does not stop us from discussing some properties that can help with spin addition. Before we dive into quantum mechanics, we can review some portions of angular momentum in classical mechanics. Like any vector quantity in classical mechanics, the sum of two angular momentum vectors does not have a magnitude equal to the sum of the magnitudes of the two individual vectors. That is to say, for two arbitrary angular momentum vectors $\vec{L}_1$ and $\vec{L}_2$ and their sum $\vec{L}_{tot}$, the magnitude of $\vec{L}_{tot}$ behaves as:

$$\left|\vec{L}_{tot}\right| \begin{cases} = \left|\vec{L}_1\right| + \left|\vec{L}_2\right| & \text{iff } \vec{L}_1 || \vec{L}_2 \\ \neq \left|\vec{L}_1\right| + \left|\vec{L}_2\right| & \text{otherwise} \end{cases}$$

Yet, while the magnitudes of vector quantities do not add, the components of the quantities do. Thus, for the same vectors,

$$L_{tot,x} = L_{1,x} + L_{2,x}$$
$$L_{tot,y} = L_{1,y} + L_{2,y}$$
$$L_{tot,z} = L_{1,z} + L_{2,z}$$

regardless of the relative directions of $\vec{L}_1$ and $\vec{L}_2$.

Perhaps unsurprisingly, these spin states share some of the same qualities as these classical vectors. That is, it is not necessarily true that the total spin quantum number of two spin states simply add, yet the total projection quantum number does. For any two arbitrary S_z eigenstates $|s_1, m_1\rangle$ and $|s_2, m_2\rangle$, the combined state $|s_{tot}, m_{tot}\rangle$ is such that

$$s_{tot} \neq s_1 + s_2$$

$$m_{tot} = m_1 + m_2$$

Yet, whereas classical vectors, when summed, give a definite vector, these quantum states do not behave as simply. Any combination of our S_z eigenstates is, instead, a linear combination of potential eigenstates, the coefficients of which are associated with the probability of the sum being in that given combined state.

It may appear daunting now, as we've implied that the states combine somewhat arbitrarily, but do not forget the rules that we have set out so far. These two conditions dictate two of the restrictions that are on s_{tot} and m_{tot} of the combined eigenstate. We can provide one more restriction on s_{tot} by again looking back at classical mechanics. By looking at these quantities, we know that the magnitude of the combined vector $\vec{L}_{tot}$ is bounded above, corresponding to when the two vectors are parallel, and bounded below, corrsponding to when the two vectors are anti-parallel. In other words,

$$\left|\vec{L}_1\right| - \left|\vec{L}_2\right| \leq \left|\vec{L}_{tot}\right| \leq \left|\vec{L}_1\right| + \left|\vec{L}_2\right|$$

where $\left|\vec{L}_1\right| > \left|\vec{L}_2\right|$. The spin quantum number works very similarly as

$$|s_1 - s_2| \leq s_{tot} \leq s_1 + s_2$$

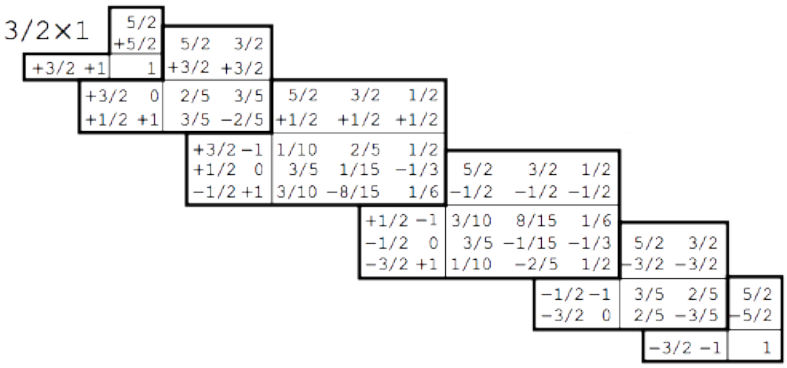
and we maintain bounds on s_{tot} while conserving m_{tot}.

So long as these two conditions are met, along with our other rules of m and s, the state should be a possibility for our spin addition. Unfortunately, finding the probabilities and such either requires an extreme amount of patience and a desire to endlessly perform algebra, or an understanding of group theory. However, thankfully, the process has been completed by a number of individuals and has since been recorded in the so called Clebsch-Gordan tables. The remainder of this spin section is devoted to learning how to read these tables.

To begin, let us determine which table to use. After all, we have been using the plural. The Clebsch-Gordan tables are divided based on the s values that are being combined. That is, if we wish to combine the arbitrary states $|s_1, m_1\rangle$ and $|s_2, m_2\rangle$ into one state $|s_{tot}, m_{tot}\rangle$, we would look at the $s_1 \times s_2$ Clebsch-Gordan table. The order of s_1 and s_2 is not important in the name of the

table and the $s_1 \times s_2$ table is the same as the $s_2 \times s_1$ table. As a result, it is common practice to have $s_1 \geq s_2$. Let us then look at an example.

Let's take one of the more complicated tables to look at here, specifically the $\frac{3}{2} \times 1$ Clebsch-Gordan table. The table has been replicated here in its most common form:



Each block of the table, denoted here by the thick black border, represents the collection of states that all have the same total projection quantum number. Each of these blocks are then subdivided into three areas: the top section representing the total spin states $|s_{tot}, m_{tot}\rangle$, the left section representing the individual projection quantum numbers m_1 and m_2, and the central states representing the relative probabilities. Within the top section, the total spin states are represented such that each column is a state where the top number is s_{tot} and the bottom number is m_{tot}. The left section is given such that each **row** gives the two projection quantum numbers of the states $|s_1, m_1\rangle$ and $|s_2, m_2\rangle$, where the leftmost number corresponds to s_1 and the the rightmost number corresponds to s_2. The central section then gives the probability of a given combination of states (corresponding to the row) being in a particular total spin state (corresponding to the column), or vice versa.

Reading the table then follows simply. For any two arbitrary states $|s_1, m_1\rangle$ and $|s_2, m_2\rangle$, we begin by finding the $s_1 \times s_2$ table. We then identify the row that has the values of m_1 and m_2 associated with our states. The corresponding row can then be read to find the probability of each of the total spin states that is possible to be in. Let us take an example.

Example 6.6. *Consider two spin states $\left|\frac{3}{2}, -\frac{1}{2}\right\rangle$ and $|1, 0\rangle$. Let us find the total spin states. We already have the $\frac{3}{2} \times 1$ table, we simply need to find the $-\frac{1}{2}, 0$ row. After identifying it, we read the total spin states at the top of the column and add in our probabilities. We can see that the combination has three potential total spin states of $\left|\frac{5}{2}, -\frac{1}{2}\right\rangle$, $\left|\frac{3}{2}, -\frac{1}{2}\right\rangle$, and $\left|\frac{1}{2}, -\frac{1}{2}\right\rangle$. The probabilities give us that these states combine as follows*

$$\left|\frac{3}{2}, 1, -\frac{1}{2}, 0\right\rangle = \sqrt{\frac{3}{5}}\left|\frac{5}{2}, -\frac{1}{2}\right\rangle - \sqrt{\frac{1}{15}}\left|\frac{3}{2}, -\frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}}\left|\frac{1}{2}, -\frac{1}{2}\right\rangle$$

where the initial state is written in the common notation of $|s_1, s_2, m_1, m_2\rangle$. Note that the square roots are included because of the how the probability of a state relates to its amplitude.

The Clebsch-Gordan table can also be used to go the opposite direction as well, with going from a total spin state to two individual spin states. In order to do this, we not only need the state, but also need the two spin quantum numbers of the individual states. Otherwise, the process is relatively similar, albeit in reverse.

Example 6.7. *Consider the spin state $\left|\frac{5}{2},\frac{1}{2}\right\rangle$ and the spin quantum numbers $s_1=\frac{3}{2}$ and $s_2=1$. We will once again use the $\frac{3}{2}\times 1$ table, but will now look at the column corresponding to our respective total spin state. Identifying the column shows us that the three potential individual states appear to be $\left|\frac{3}{2},1,\frac{3}{2},-1\right\rangle$, $\left|\frac{3}{2},1,\frac{1}{2},0\right\rangle$, and $\left|\frac{3}{2},1-\frac{1}{2},1\right\rangle$, using the notation given in Example 6.6. The resulting dissolution gives*

$$\left|\frac{5}{2},\frac{1}{2}\right\rangle=\sqrt{\frac{1}{10}}\left|\frac{3}{2},1,\frac{3}{2},-1\right\rangle+\sqrt{\frac{3}{5}}\left|\frac{3}{2},1,\frac{1}{2},0\right\rangle+\sqrt{\frac{3}{10}}\left|\frac{3}{2},1,-\frac{1}{2},1\right\rangle$$

Note that since the two individual spin quantum numbers must also be given, it is sometimes given that the total spin state is of the form $|s_1,s_2;s_{tot},m_{tot}\rangle$.

This is the simple matter of reading a Clebsch-Gordan table. As we have stated, the intricacies of this is steeped in group theory, and the true nature of the Clebsch-Gordan coefficients, as they are aptly named, is found via methods from this practice. However, if one is especially keen or willing, they can derive these coefficients through the properties of the spin operators and their respective ladder operators. Doing so is tedious, and a simple example will be given here, but, more importantly, it only secures one table of coefficients at a time and so the full description of the Clebsch-Gordan coefficients is left out. Finding the coefficients for certain combinations of s_1 and s_2 is, however, still a common practice problem given in upper year or even graduate level quantum mechanics courses.

Example 6.8. *For the following example, we will be exploring the addition of two spin-$\frac{1}{2}$ particles. That is, we will be observing $s_1=\frac{1}{2}$ and $s_2=\frac{1}{2}$. Furthermore, we will take the common notation standard of the two particle's state being written as $|s_1,s_2,m_1,m_2\rangle$ and the total spin state being written as $|s_1,s_2;s_{tot},m_{tot}\rangle$. To begin this exploration, we will start with the spin state $\left|\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle$, or rather the two spin particles each being in the "spin-up" configuration. Recall our conditions of s_{tot} and m_{tot}. That is that the total projection is conserved*

$$m_{tot}=m_1+m_2$$

and that the total spin is bounded

$$|s_1-s_2|\leq s_{tot}\leq|s_1+s_2|$$

These two conditions tell us that the possible s_{tot} values are 0 and 1 and that for the given state, with the two spin particles being in the spin-up configuration, the total projection is $m_{tot}=1$. Since m_{tot} cannot be equal to 1 if $s_{tot}=0$, we only have one possibility for the given two spin particle state. As a result

$$\left|\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle=\left|\frac{1}{2},\frac{1}{2};1,1\right\rangle$$

The same logic can be applied to find

$$\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\rangle=\left|\frac{1}{2},\frac{1}{2};1,-1\right\rangle$$

From here, we can apply some ladder operators. Suppose we increase or decrease the total projection of our states. As we have previously stated, we need our total projection to be conserved. However, we cannot increase or decrease the projection of each of the two particles by half a quanta, doing so is prohibited by quantum mechanics. We also cannot do so be two, as if we decrease one of our particles by two, we would have to increase the other by one which is forbidden since m is bounded

by s (or the same with increasing if looking at the all spin-down state). Thus, if we decrease the projection of the total state, we can either decrease the projection m_1 and do nothing to m_2 or vice versa. To use our operators, let us first define operators by which particle they act on. Let $S_+^{(1)}$ and $S_-^{(1)}$ be the ladder operators for s_1 and m_1 and $S_+^{(2)}$ and $S_-^{(2)}$ be the ladder operators for s_2 and m_2. Let $S_+^{(tot)}$ and $S_-^{(tot)}$ be the ladder operators for s_{tot} and m_{tot}. Finally, let $I^{(1)}$ and $I^{(2)}$ be the identity operators for s_1 and s_2, respectively. Then we can write the total ladder operators in terms of the individual ladder operators

$$S_+^{(tot)} = S_+^{(1)} I^{(2)} + I^{(1)} S_+^{(2)}$$

$$S_-^{(tot)} = S_-^{(1)} I^{(2)} + I^{(1)} S_-^{(2)}$$

Applying this to our previous states gives us:

$$\begin{aligned} S_-^{(tot)} \left|\frac{1}{2},\frac{1}{2};1,1\right\rangle = \left|\frac{1}{2},\frac{1}{2};1,0\right\rangle &= \left(S_-^{(1)} I^{(2)} + I^{(1)} S_-^{(2)}\right) \left|\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle \\ &= \left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle + \left|\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle \end{aligned}$$

Keeping in mind that our state needs to be normalized then gives us the following:

$$\left|\frac{1}{2},\frac{1}{2};1,0\right\rangle = \sqrt{\frac{1}{2}}\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle + \sqrt{\frac{1}{2}}\left|\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle$$

which is unique up to a phase. Following our previous phase conventions, we have picked this phase such that both of our coefficients are positive and real. This then gives us four of our six coefficients. Finding the last two requires a bit more ingenuity. First, we can note that each of these coefficients go either way. That is to say that $\langle s_1, s_2, m_1, m_2 \,|s_1, s_2; s_{tot}, m_{tot}\rangle = \langle s_1, s_2; s_{tot}, m_{tot} \,|s_1, s_2, m_1, m_2\rangle$. As such, we have

$$\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{1}{2}}\left|\frac{1}{2},\frac{1}{2};1,0\right\rangle + C_1\left|\frac{1}{2},\frac{1}{2};0,0\right\rangle$$

$$\left|\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle = \sqrt{\frac{1}{2}}\left|\frac{1}{2},\frac{1}{2};1,0\right\rangle + C_2\left|\frac{1}{2},\frac{1}{2};0,0\right\rangle$$

Now, we have written all possible states here and we must have the our state is normalized. Thus, we can easily see that $|C_1| = |C_2| = \frac{1}{2}$. However, we run into a problem if we make both of these coefficients $\frac{1}{2}$. Namely that we do not maintain orthogonality and we see that

$$\left\langle\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\middle|\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle = 1$$

which is quite a problem. We cannot change the magnitudes as they are the only way our equations currently work out. However, we have one degree of freedom which is phase. In order to achieve this, we must have either C_1 or C_2 be $-\sqrt{\frac{1}{2}}$. It is recommended that one verifies this in their own time using our choice of coefficients for $\left|\frac{1}{2},\frac{1}{2};1,0\right\rangle$. As such, let us take $C_2 = -\frac{1}{2}$, another common convention that is used. Once we have this, all of our coefficients are given

$$\left|\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle = \left|\frac{1}{2},\frac{1}{2};1,1\right\rangle$$

$$\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{1}{2}}\left|\frac{1}{2},\frac{1}{2};1,0\right\rangle + \sqrt{\frac{1}{2}}\left|\frac{1}{2},\frac{1}{2};0,0\right\rangle$$

$$\left|\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle = \sqrt{\frac{1}{2}}\left|\frac{1}{2},\frac{1}{2};1,0\right\rangle - \sqrt{\frac{1}{2}}\left|\frac{1}{2},\frac{1}{2};0,0\right\rangle$$

$$\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\rangle = \left|\frac{1}{2},\frac{1}{2};1,-1\right\rangle$$

And the table for $\frac{1}{2} \times \frac{1}{2}$ may be completed.